1. (10 pts) Let $P, Q, R$ be statements. Show, using truth tables, that the statements $(P \text{ and } Q) \Rightarrow R$ and $P \Rightarrow (Q \Rightarrow R)$ are equivalent.

2. (a) (4 pts) Give a proof or counterexample for the following statement: For all integers $m$ there exists an integer $n$ such that $mn$ is even.

(b) (6 pts) Prove that among two consecutive integers $n$ and $n + 1$ at least one is odd.

3. Let $A, B, C$ be sets.
   
   (a) (7 pts) Show that $A \cap (B \cup C) \subseteq (A \cap B) \cup C$.

   (b) (3 pts) Show that the equality $A \cap (B \cup C) = (A \cap B) \cup C$ does not hold in general by giving an example of three sets $A, B, C$ for which the equality fails.

4. (10 pts) Prove that 4 divides $3^{2n+1} + 1$ for all positive integers $n$. 
1. The truth table is as follows.

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<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$R$</th>
<th>$(P \text{ and } Q) \Rightarrow R$</th>
<th>$Q \Rightarrow R$</th>
<th>$P \Rightarrow (Q \Rightarrow R)$</th>
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We see that the columns corresponding to $(P \text{ and } Q) \Rightarrow R$ and $P \Rightarrow (Q \Rightarrow R)$ are identical and hence these statements are equivalent.

2. 
   (a) For all $m$ we can take $n = 2$ so that $mn = 2m$ which is even, so the statement is true.

   (b) We argue by contradiction. Suppose that both $n$ and $n + 1$ are even. Then we have $n = 2a$ and $n + 1 = 2b$ for some integers $a, b$. Thus
   
   $$1 = n + 1 - n = 2a - 2b = 2(a - b)$$

   Then we must have $a - b > 0$ and since this is an integer it follows that $a - b \geq 1$. But this implies that $1 = 2(a - b) \geq 2$, which is a contradiction.

3. 
   (a) Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B$ or $x \in C$.
   
   If $x \in B$, we have that since also $x \in A$, $x \in A \cap B$ and thus $x \in (A \cap B) \cup C$.
   
   If $x \in C$, then $x \in (A \cap B) \cup C$ as well.
   
   In all cases, we have shown that $x \in (A \cap B) \cup C$ and therefore
   
   $$A \cap (B \cup C) \subseteq (A \cap B) \cup C$$

   (b) Let $A = \{1\}$, $B = \{1\}$, $C = \{2\}$. Then $A \cap (B \cup C) = \{1\}$, while $(A \cap B) \cup C = \{1, 2\}$, so the two sets are not equal in this case.

4. We use induction on $n = 1$.

   **Base case:** For $n = 1$, we have $3^{2n+1} + 1 = 28 = 4 \cdot 7$, a multiple of 4.

   **Inductive step:** Suppose that for some integer $k \geq 1$, 4 divides $3^{2k+1} + 1$, thus we have $3^{2k+1} + 1 = 4q$ for some integer $q$ and so $3^{2k+1} = 4q - 1$. Then we have
   
   $$3^{2(k+1)+1} + 1 = 3^{2k+3} + 1 = 9 \cdot 3^{2k+1} + 1 = 9(4q - 1) + 1 = 36q - 8 = 4(9q - 2)$$

   and thus 4 divides $3^{2(k+1)+1} + 1$, as we want.