MATH 109 WINTER 2019 MIDTERM II PRACTICE QUESTIONS

The following questions are meant to help you prepare for the exam. However, you should still review all the homework problems, lecture notes and Chapters 8-16 of the textbook as well.

1. Let $f : X \to Y$ and $g : Y \to Z$ be functions and $h = g \circ f$.
   
   (1) Show that if $f$ and $g$ are injective, then $h$ is injective.
   
   (2) Show that if $f$ and $g$ are surjective, then $h$ is surjective.
   
   (3) Show that if $h$ is injective, then $f$ is injective. Show that the converse is not true by demonstrating an explicit example with $f$ injective and $h$ not injective.
   
   (4) Show that if $h$ is surjective, then $g$ is surjective. Show that the converse is not true by demonstrating an explicit example with $g$ surjective and $h$ not surjective.

2. Let $\mathbb{R}^\times = \{x \in \mathbb{R} \mid x \neq 0\}$ be the set of non-zero real numbers. Consider the function $f : \mathbb{R}^\times \to \mathbb{R}^\times$ defined by $f(x) = \frac{3}{x}$.
   
   (1) Show that $f$ is bijective.
   
   (2) Find its inverse function $f^{-1} : \mathbb{R}^\times \to \mathbb{R}^\times$.

3. Let $A$ be a non-empty set. Show that if $A \times A$ is countable, then $A$ is countable. Deduce that $\mathbb{R}^2$ is uncountable.

4. Find the greatest common divisor of 126 and 312.

5. Let $n, r$ be integers with $0 \leq r \leq n$. For $r + 1 \leq i \leq n + 1$, let
   
   $$X_i = \{A \in \mathcal{P}_{r+1}(\mathbb{N}_{n+1}) \mid i \text{ is the largest element of } A\}$$
   
   Show that
   
   $$\bigcup_{i=r+1}^{n+1} X_i = \mathcal{P}_{r+1}(\mathbb{N}_{n+1})$$
   
   Deduce that for all integers $0 \leq r \leq n$
   
   $$\sum_{i=r}^{n} \binom{i}{r} = \binom{r+1}{r} + \binom{r+1}{r+1} + \ldots + \binom{n}{r} = \binom{n+1}{r+1}$$
   
   Now prove this identity using induction on $n$.

6. Let $n$ be a positive integer. Show, by differentiating $(1+x)^n$ with respect to $x$, that
   
   $$\sum_{i=0}^{n} i \binom{n}{i} = \binom{n}{1} + 2 \binom{n}{2} + \ldots + n \binom{n}{n} = n \cdot 2^{n-1}$$

7. Consider a set $X$ consisting of 10 distinct numbers chosen from the set $\{1, 2, 3, \ldots, 40\}$ of the first 40 natural numbers. Show that $X$ must contain two subsets $Y$ and $Z$ with $Y \neq Z$ and $|Y| = |Z| = 3$, such that the sum of the elements in $Y$ is the same as the sum of the elements in $Z$.

8. Show that the union of two countable sets is countable.
9. A set of 100 voters are asked about how they plan to vote on three statewide ballot propositions, Propositions A, B, and C. On each proposition, they can vote either yes or no.

In the survey, 40 said they plan to vote yes on A, 50 said they will vote yes on B, and 60 said they will vote yes on C. 30 voters plan to vote for both A and B, 10 plan to vote for both A and C, 5 voters plan to vote yes on all three propositions, and 5 voters plan to vote no on all three propositions.

How many voters are there that are planning to vote yes on B but no on both A and C?

10. Prove that 3 divides \( n^3 - n \) for all integers \( n \).

11. The set \( \mathbb{Q}[x] \) of polynomials with rational coefficients consists of functions of the form

\[
a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0
\]

where \( a_n, \ldots, a_0 \in \mathbb{Q} \) and \( a_n \neq 0 \). \( n \) (which can vary) is called the degree of the polynomial.

Show that \( \mathbb{Q}[x] \) is denumerable. You may use the fact that a denumerable union of denumerable sets is denumerable.

12. Let \( n \) be an integer.

   (1) Prove that there does not exist an integer \( q \) such that \( n^2 = 6q + 5 \).

   (2) Show that 1205 is not a square.

13. (1) Show that 6 divides an integer \( n \) if and only if 6 divides \( n^2 \).

   (2) Show that \( \sqrt{6} \) is not rational.