7.5
Suppose there exist \( q \in \mathbb{Z} \) such that \( n = 2q + 1 \). Let \( p = 2q^2 + 2q \), then \( 2p + 1 = 2(2q^2 + 2q) + 1 = (2q + 1)^2 = n^2 \). So we have found a \( p \) such that \( n^2 = 2p + 1 \). So we are done.

7.7
Suppose \((x, y) \in (A \times B) \cup (C \times D)\), then either (1) \((x, y) \in (A \times B)\) or (2) \((x, y) \in (C \times D)\). If \((x, y) \in (A \times B)\), then \( x \in A \) and \( y \in B \). Then \( x \in (A \cup C) \) and \( y \in (B \cup D) \), so \((x, y) \in (A \cup C) \times (B \cup D)\). We can use a similar reasoning to argue that if \((x, y) \in (C \times D)\) then \((x, y) \in (A \cup C) \times (B \cup D)\).

Example \( A = \{a\}, B = \{b\}, C = \{c\}, D = \{d\} \). Then \((a, d) \in (A \cup C) \times (B \cup D)\), but \((a, d) \notin (A \times B) \cup (C \times D)\).

8.1
For \((x, y) \in \mathbb{R}^2\), we know either \( x > y \), \( x = y \) or \( x < y \). When \( x > y \) or \( x < y \), there is no ambiguity in the definition of \( g(x, y) \). So we only have to worry about \( x = y \), as we could use either of the two definitions here. The case of \( x = y \), however, the fist definition gives \( x \) and the second gives \( y \), so the two definitions agree. Therefore, \( g \) is well-defined.

Suppose \( x \geq y \). Then \( g(x, y) = x \) and \(|x - y| = x - y\), so

\[
  f(x, y) = \frac{x + y}{2} + \frac{x - y}{2} = x.
\]

Now suppose \( x \leq y \). Then \( g(x, y) = y \) and \(|x - y| = y - x\), so

\[
  f(x, y) = \frac{x + y}{2} + \frac{y - x}{2} = y.
\]

Since \( f(x, y) = g(x, y) \) for all \((x, y) \in \mathbb{R}^2\), \( f = g \).

8.3
There are many possible solutions.

(i) \( f(x) = x \)

(ii) \( f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \)

(iii) \( f(x) = \begin{cases} x & \text{if } x \notin \mathbb{Z} \\ x + 1/2 & \text{otherwise} \end{cases} \)

(iv) The floor function: \( f(x) = n \in \mathbb{Z} \) if \( n \leq x < n + 1 \)
8.5

(i)

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>z</td>
</tr>
<tr>
<td>b</td>
<td>y</td>
</tr>
<tr>
<td>c</td>
<td>z</td>
</tr>
<tr>
<td>d</td>
<td>x</td>
</tr>
</tbody>
</table>

(ii) Not a graph. No element specified for the value at c.
(iii) Not a graph. Two elements specified for the value at b.

(iv)

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>y</td>
</tr>
<tr>
<td>b</td>
<td>z</td>
</tr>
<tr>
<td>c</td>
<td>w</td>
</tr>
<tr>
<td>d</td>
<td>x</td>
</tr>
</tbody>
</table>

II.5

(i)

<table>
<thead>
<tr>
<th>x ∈ A</th>
<th>x ∈ B</th>
<th>x ∈ C</th>
<th>x ∈ A ∪ C</th>
<th>x ∈ A ∪ C − B</th>
<th>x ∈ (A − B) ∪ C</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Since for every T in the second to last column, I have a corresponding T in the last column, so the statement is true.

For part (ii), we can construct a similar truth tale with x ∈ A, x ∈ B, x ∈ C, x ∈ (A ∩ C) − B and x ∈ (A − B) ∩ C as the columns and check that entries for the x ∈ (A ∩ C) − B and x ∈ (A − B) ∩ C columns match.

Suppose A ∪ C − B = (A − B) ∪ C, by looking at the truth tables, we know that there must be no x such that either x ∈ A, B, C and x /∈ A, x ∈ B, C. So we there must be no x that is in B and C. So B ∪ C = ∅.

Now suppose B ∩ C = ∅, then we can remove the the 5th and first column of our truth table. After removing those columns, x ∈ A ∪ C − B and x ∈ (A − B) ∪ C will have identical entries in the truth table. So B ∩ C = ∅ ⇒ (A ∪ C) − B = (A − B) ∪ C.

II.14

\( f \circ f = x^4, f \circ g = (x^2 - 1)^2 \), \( g \circ f = x^4 - 1 \), \( g \circ g = (x^2 - 1)^2 - 1 \).

We want to solve

\[ gf(x) = x^4 - 1 = (x^2 - 1)^2 = fg(x). \]

Simplifying gives:

\[ x^4 - 1 = x^4 - 2x^2 + 1 \]

and so we have

\[ x^2 = 1 \]

so \( x = -1, 1 \). Therefore \( \{ x \in \mathbb{R} \mid fg(x) = gf(x) \} = \{-1, 1\} \).
II.15
We can do this case by case. Let us call the new function $f(x) = \chi_A(x)\chi_B(x)$.

(i) Suppose $x \in A$ and $x \in B$, then $\chi_A(x) = 1$, $\chi_B(x) = 1$, so $x$ gets sent to 1 under the function. So for such $x$, $f(x) = \chi_{A \cap B}(x)$.

(ii) Suppose $x \in A$ and $x \notin B$, then $\chi_A(x) = 1$, $\chi_B(x) = 0$, so $x$ gets sent to 0 under the function. Since $x \notin B$, means $x \notin A \cap B$. So for such $x$, $f(x) = \chi_{A \cap B}(x)$.

(iii) Similarly, an argument can be made for the case $x \notin A$ and $x \in B$.

(iv) Similarly, an argument can be made for the case $x \notin A$ and $x \notin B$.

So we have just shown that $f = \chi_{A \cap B}$.

By what we just showed, we know $\chi_C(x) = \chi_A(x) + \chi_B(x) - \chi_{(A \cap B)}(x)$

We want to show $C = A \cup B$. To prove this, we use a similar process as before by breaking down into cases of $x \in A, x \notin A, x \in B, x \notin B$, and show that $\chi_A(x) + \chi_B(x) - \chi_{(A \cap B)}(x) = \chi_{A \cup B}(x)$ for all $x$.

II.16

(i) bijective. $f^{-1}(x) = x + 1$.

(ii) bijective. $f^{-1}(x) = \sqrt{x}$.

(iii) surjective, not injective

(iv) Notice $f(x) = (x - 1)^3$. It is bijective. $f^{-1}(x) = \sqrt[3]{x} + 1$

(v) injective, not surjective

(vi) bijective, the inverse is defined as:

$$f^{-1}(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -\sqrt{-x} & \text{if } x < 0 \end{cases}$$

II.18

Let $z \in \mathbb{Z}$. Since $g : Y \to \mathbb{Z}$ is surjective, there exists $y \in Y$ such that $g(y) = z$. Since $f : X \to Y$ is surjective, there exists $x \in X$ such that $f(x) = y$. So we have $x \in X$ such that $g \circ f(x) = z$. So $g \circ f$ is surjective.

II.19

Suppose there exists $g : Y \to X$ such that $f \circ g = I_Y$. Then for all $y \in Y$, we have $f(g(y)) = f \circ g(y) = I_Y(y) = y$. So $g(y)$ is a preimage of $y$. So $f$ is surjective.

Now suppose $f$ is surjective. Then for all $y \in Y$, there exist $x$ such that $f(x) = y$. Define a function $g : Y \to X$ by letting $g(y) = x$. Then, for all $y$, we have

$$f \circ g(y) = f(x) = y = I_Y(y).$$

So we have check $f \circ g = I_Y$ for the function $g$ we just defined.