MATH 194, HOMEWORK 6, DUE IN CLASS FRIDAY, JUNE 4

We use the notation for the finite market model from Williams' notes, Chapter 3. We assume that the market is viable, and let P^* denote a risk-neutral probability equivalent to P. Let $M_t := \frac{dP_t^*}{dP_t}$ for $t = 0, 1, \ldots, T$. It was shown in class that the return R_1^i on the i^{th} on stock i during the first period, defined by $R^i := \frac{S_1^i - S_0^i}{S_0^i}$. has a mean (with respect to P) $\bar{R}^i$ satisfying $\bar{R}^i - r = -\operatorname{Cov}(R^i, M_1)$, where Cov is computed with respect to P. Let R denote the return on a fixed portfolio during the first period, in which the portfolio was determined by $\phi_1 := \{\phi_1^0, \ldots, \phi_1^d\}$ in the respective securities. (We may assume these are $\mathcal{F}_0$-measurable, hence constant.) By this we mean

$$R := \frac{V_1 - V_0}{V_0}.$$

We showed in class that $\bar{R} - r = -\operatorname{Cov}(R, M_1)$.

1. (a) Suppose the European contingent claim with expiration $T = 1$ and payoff M_1 is attainable. Show that the ECC with payoff $a + bM_1$ is also attainable for any scalars a, b.

(b) Under the hypotheses of (a), fix scalars a and $b \neq 0$, and let ϕ' denote a trading strategy with value V' satisfying $V_1' = a + bM_1$. Let R' denote the corresponding return. Show using properties of covariance that

$$\operatorname{Cov}[R, M_1) = \frac{V_0'}{b} \operatorname{Cov}(R, R').$$

Use this to derive the formula

$$\bar{R} - r = -\frac{V_0'}{b} \operatorname{Cov}(R, R').$$

The special case $\phi = \phi'$ yields $\bar{R} - r = -\frac{V_0'}{b} \operatorname{Var}(R')$. Hence obtain

$$\bar{R} - r = \frac{\operatorname{Cov}(R, R')}{\operatorname{Var}(R')} (\bar{R}' - r).$$

(This says that the risk premium associated with ϕ is proportional to the risk premium corresponding to another strategy ϕ', with constant of proportionality $\beta := \frac{\operatorname{Cov}(R,R')}{\operatorname{Var}(R')}$.)

2. Let ξ_1 through ξ_T denote iid random variables, each taking two possible values d, u, with $0 < d < u$, and let $p := P\{\xi_j = u\}$. Use the multiplicative form of the Central Limit Theorem to identify the approximate distribution of $\xi_1 \xi_2 \ldots \xi_T$, assuming T is large, and that the interest rate r per time period is small, and that $d < 1 + r < u$ with u and d so close to 1 that we may approximate $\log u$ by $u - 1$ and $\log d$ by $d - 1$.

3. Exercise 3, section 3.7 of Williams' notes.

4. Exercise 5 (a), section 2.4 of Williams' notes.