Do not turn the page until instructed to begin.

**Turn off and put away your cell phone.**

No calculators or any other devices are allowed.  
You may use one 8.5×11 page of handwritten notes, but no other assistance.  
Read each question carefully, answer each question completely, & show all of your work.  
Write your solutions clearly and legibly; no credit will be given for illegible solutions.  
If any question is not clear, ask for clarification.  
Good luck!

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1. Take the following indefinite integrals.

(a) (5 points) \[ \int \frac{(2-x)^2}{x} \, dx. \] (hint: FOIL and separate terms)

\[
\int \frac{(2-x)^2}{x} \, dx = \int \frac{4 - 4x + x^2}{x} \, dx \\
= \int \left( \frac{4}{x} - \frac{4x}{x} + \frac{x^2}{x} \right) \, dx \\
= 4 \int \frac{1}{x} \, dx - 4 \int \, dx + \int x \, dx \\
= 4 \ln |x| - 4x + \frac{x^2}{2} + C
\]

(b) (5 points) \[ \int \frac{x}{(2-x)^2} \, dx. \] (hint: u-substitution)

\[
u = 2 - x \implies \frac{du}{dx} = -1 \implies dx = -du.
\]

Using substitutions, we now have

\[
\int \frac{x}{(2-x)^2} \, dx = - \int \frac{x}{u^2} \, du.
\]

We still need to replace the \( x \) in the last integral in order to put everything in terms of \( u \) so we can take the integral. Luckily we can solve \( u = 2 - x \) for \( x \) to get \( x = 2 - u \), and now we have

\[
\int \frac{x}{(2-x)^2} \, dx = - \int \frac{2 - u}{u^2} \, du \\
= - \int \left( \frac{2}{u^2} - \frac{u}{u^2} \right) \, du \\
= -2 \int \frac{1}{u^2} \, du + \int \frac{1}{u} \, du \\
= -2 \frac{u^{-1}}{-1} + \ln |u| + C \\
= \frac{2}{u} + \ln |u| + C \\
= \frac{2}{2-x} + \ln |2-x| + C
\]
2. (10 points) A company has a continuous income stream of \( P(t) = 10 + t \) million dollars per year. Assuming this income is directly deposited in an account making 5\% interest per year, how much money will the company have in 2 years?

The formula for the future value of a continuous income stream is

\[
\text{Future value} = \int_0^M P(t)e^{r(M-t)} \, dt,
\]

where \( M = 2 \) and \( r = 0.05 \) (the interest rate). Substituting in \( P(t) = 10 + t \), we get

\[
\text{Future value} = \int_0^2 (10 + t)e^{r(2-t)} \, dt = \int_0^2 10e^{r(2-t)} \, dt + \int_0^2 te^{r(2-t)} \, dt.
\]

Let’s calculate these last two integrals separately and then add them to get the final answer. The first integral requires \( u \)-substitution with \( u = r(2-t) \), which gives us \( \frac{du}{dt} = -r \), or in other words \( dt = \frac{-1}{r} \, du \), and substituting this in we get

\[
\int_0^2 10e^{r(2-t)} \, dt = \int 10e^u \left( \frac{-1}{r} \right) \, du = 10 \left( \frac{-1}{r} \right) \int e^u \, du = 10 \left( \frac{-1}{r} \right) \left( e^{r(2-t)} \right)_0^2 = 10 \left( \frac{-1}{r} \right) \left( e^{(2-2)} - e^{(2-0)} \right) = 10 \left( \frac{-1}{r} \right) \left( 1 - e^{2r} \right) = -200 \left( 1 - e^{1/r} \right) \approx 21.03.
\]

The second integral involves integration by parts.

\[
\begin{align*}
\int te^{r(2-t)} \, dt &= \int \frac{u}{v} e^{r(2-t)} \, du - \int \frac{v}{u} e^{r(2-t)} \, dv \\
&= \int u \frac{1}{r} e^{r(2-t)} \, dt - \int v e^{r(2-t)} \, du \\
&= \left. \frac{t}{r} e^{r(2-t)} \right|_0^2 - \left. \frac{1}{r^2} e^{r(2-t)} \right|_0^2 \\
&= -\frac{4}{r} e^{2-r} - \frac{1}{r^2} e^{r(2-0)} + C \\
&= -\frac{4}{r} + \frac{1}{r^2} e^{2r} \\
&= -40 - 400 + 400 e^{1/10} \approx 2.07.
\end{align*}
\]

So the answer is about 23.10 million dollars.
3. (10 points) Find the shaded area between the curves depicted below.

The curves intersect when $\sin(2x) = \frac{1}{2}$. The two angles which have a sine of $\frac{1}{2}$ are $\pi/6$ and $5\pi/6$, so setting $2x$ equal to these angles and solving for $x$ gives us $x = \pi/12$ and $x = 5\pi/12$. The area between the curves is

$$\int_{\pi/12}^{5\pi/12} (\sin(2x) - \frac{1}{2}) \, dx = \int_{\pi/12}^{5\pi/12} \sin(2x) \, dx - \int_{\pi/12}^{5\pi/12} \frac{1}{2} \, dx$$

$$= -\frac{1}{2} \cos(2x) \bigg|_{\pi/12}^{5\pi/12} - \frac{1}{2}x \bigg|_{\pi/12}^{5\pi/12}$$

$$= \left( -\frac{1}{2} \cos(5\pi/6) + \frac{1}{2} \cos(\pi/6) \right) - \left( \frac{5\pi}{24} - \frac{\pi}{12} \right)$$

$$= \left( -\frac{1}{2} \left( -\frac{\sqrt{3}}{2} \right) + \frac{1}{2} \left( \frac{\sqrt{3}}{2} \right) \right) - \left( \frac{5\pi}{24} - \frac{\pi}{12} \right)$$

$$= \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$
4. Find more indefinite integrals.

(a) (5 points) \[ \int \frac{1}{1 + 9x^2} \, dx. \] (hint: trig. substitution)

Let \( x = \frac{1}{3} \tan \theta \). Then \( dx = \frac{1}{3} \sec^2 \theta \, d\theta \). Using the trigonometric identity for the tangent: \( 1 + \tan^2 \theta = \sec^2 \theta \), we have

\[
\int \frac{1}{1 + 9x^2} \, dx = \int \frac{1}{1 + \tan^2 \theta} \, dx \\
= \int \frac{1}{\sec^2 \theta} \frac{1}{3} \sec^2 \theta \, d\theta \\
= \int \frac{1}{3} \, d\theta \\
= \frac{1}{3} \theta + C \\
= \frac{1}{3} \arctan(3x) + C.
\]

(b) (5 points) \[ \int \frac{18x}{1 + 9x^2} \, dx. \] (hint: u-substitution)

Let \( u = 1 + 9x^2 \). Then \( du = 18x \, dx \), and

\[
\int \frac{18x}{1 + 9x^2} \, dx = \int \frac{1}{u} \, du \\
= \ln |u| + C \\
= \ln(1 + 9x^2) + C.
\]
5. (10 points) Find the volume of the spool depicted below with radius given by the curve \( r = 1 + x^2 \) for \(-1 \leq x \leq 1\).

The volume formula for a volume of revolution is

\[
\text{Volume} = \pi \int (r(x))^2 \, dx,
\]

where \( r(x) \) is the radius function. So in this case we have

\[
\text{Volume} = \pi \int_{-1}^{1} (1 + x^2)^2 \, dx
\]

\[
= \pi \int_{-1}^{1} 1 + 2x^2 + x^4 \, dx
\]

\[
= \pi \left( x + \frac{2}{3}x^3 + \frac{1}{5}x^5 \right)\Bigg|_{-1}^{1}
\]

\[
= \pi \left( (1 + \frac{2}{3} + \frac{1}{5}) - (\frac{2}{3} - \frac{1}{5}) \right) = \frac{14}{15}\pi.
\]
6. Take the following indefinite integrals.

(a) (5 points) \( \int \frac{\ln(x)}{x} \, dx \).  
(hint: u-substitution)

Letting \( u = \ln(x) \) gives us \( \frac{du}{dx} = \frac{1}{x} \), or \( du = \frac{1}{x} \, dx \).

\[
\int \frac{\ln(x)}{x} \, dx = \int u \, du = \frac{u^2}{2} + C = \frac{(\ln(x))^2}{2} + C
\]

(b) (5 points) \( \int \frac{2}{x^2 - x} \, dx \).  
(hint: partial fractions)

We factor \( x^2 - x \) on the bottom into \( x(x - 1) \), so the general form for the partial fraction expansion should be

\[
\frac{2}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}.
\]

To add the fractions on the right hand side, we need to make the denominators the same

\[
\frac{2}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} = \frac{A(x-1) + Bx}{x(x-1)} = \frac{(A + B)x - A}{x(x-1)}.
\]

From here, we need the top to be equal to \((0)x + 2\), so we get the two equations

\[ A + B = 0 \quad \text{and} \quad -A = 2, \]

so \( A = -2 \) and \( B = 2 \). Then the original integral can be calculated.

\[
\int \frac{2}{x^2 - x} \, dx = \int \left( \frac{-2}{x} + \frac{2}{x-1} \right) \, dx = \frac{-2 \ln |x| + 2 \ln |x-1| + C}{x}.
\]
7. Solve the following initial value problem.

\[ \frac{dy}{dx} = x \sin(x) \quad y(0) = 1. \]

Integrate both sides of the differential equation to get \( y(x) = \int x \sin(x) \, dx \). This integral can be solved with integration by parts.

\[
\begin{align*}
  u &= x & \Rightarrow & & du = dx \\
  dv &= \sin(x) \, dx & \Rightarrow & & v = -\cos(x) \\
  \int u \, dv &= uv - \int v \, du \\
  y(x) &= \int x \sin(x) \, dx = -x \cos(x) + \int \cos(x) \, dx \\
  &= -x \cos(x) + \sin(x) + C
\end{align*}
\]

Therefore \( y(x) = -x \cos(x) + \sin(x) + C \). Using the initial condition \( y(0) = 1 \), we have

\[
1 = y(0) = -0 \cos(0) + \sin(0) + C = 0 + 0 + C \quad \Rightarrow \quad C = 1.
\]

So the final answer is \( y(x) = -x \cos(x) + \sin(x) + 1 \).
8. (10 points) You’re drinking coffee while working on a practice final, but you’re so concentrated on the questions that you forget to continue drinking. Your coffee starts at 150°F and the temperature of the room is 75°F. According to Newton’s Law of Cooling, the coffee’s temperature $y(t)$ (in degrees Fahrenheit) as a function of time (in minutes) is given by the differential equation
\[
\frac{dy}{dt} = \frac{75 - y}{50}, \quad y(0) = 150,
\]

How long until the coffee reaches 100°F?

This is a separable differential equation. It can be rewritten as
\[
\frac{1}{(75 - y)} \, dy = \frac{1}{50} \, dt,
\]

and integrating both sides gives us
\[
- \ln(75 - y) = \frac{t}{50} + C.
\]

We now solve for $y$ in terms of $t$.
\[
\ln(75 - y) = -\frac{t}{50} - C
\]
\[
75 - y = e^{-\frac{t}{50}} \cdot e^{-C}
\]
\[
75 - y = C_0 e^{-\frac{t}{50}} \quad \text{(let } e^{-C} = C_0)\]
\[
75 - y = C_0 e^{-\frac{t}{50}}
\]
\[
y(t) = 75 - C_0 e^{-\frac{t}{50}},
\]

The initial condition is $y(0) = 150$. Plugging this in we get
\[
150 = y(0) = 75 - C_0 e^{-\frac{0}{50}} = 75 - C_0 \quad \implies \quad C_0 = -75.
\]

Therefore the temperature of the coffee is $y(t) = 75 + 75e^{-t/50}$. To answer the question of how long it takes for the coffee to reach 100°F, we write the equation $y(t) = 100$ and solve for $t$.
\[
75 + 75e^{-t/50} = 100
\]
\[
75e^{-t/50} = 25
\]
\[
e^{-t/50} = \frac{1}{3}
\]
\[
-t/50 = \ln(\frac{1}{3})
\]
\[
t = -50 \ln(\frac{1}{3}) \quad (\approx 23.86 \text{ minutes})
\]
### Table of Integrals

#### Basic Functions

1. \[ \int x^n \, dx = \frac{1}{n+1} x^{n+1} + C \quad \text{(if } n \neq 1) \]
2. \[ \int \frac{1}{x} \, dx = \ln |x| + C \]
3. \[ \int a^x \, dx = \frac{1}{\ln(a)} a^x + C \quad \text{(if } a > 0) \]
4. \[ \int \ln(x) \, dx = x \ln(x) - x + C \]
5. \[ \int \sin(x) \, dx = -\cos(x) + C \]
6. \[ \int \cos(x) \, dx = \sin(x) + C \]
7. \[ \int \tan(x) \, dx = -\ln |\cos(x)| + C \]

#### Products of \(e^x, \cos(x), \sin(x)\)

8. \[ \int e^{ax} \sin(bx) \, dx = \frac{1}{a^2 + b^2} e^{ax} \left[ a \sin(bx) - b \cos(bx) \right] + C \]
9. \[ \int e^{ax} \cos(bx) \, dx = \frac{1}{a^2 + b^2} e^{ax} \left[ a \cos(bx) + b \sin(bx) \right] + C \]
10. \[ \int \sin(ax) \sin(bx) \, dx = \frac{1}{b^2 - a^2} \left[ a \cos(ax) \sin(bx) - b \sin(ax) \cos(bx) \right] + C \quad \text{(if } a \neq b) \]
11. \[ \int \cos(ax) \cos(bx) \, dx = \frac{1}{b^2 - a^2} \left[ b \cos(ax) \sin(bx) - a \sin(ax) \cos(bx) \right] + C \quad \text{(if } a \neq b) \]
12. \[ \int \sin(ax) \cos(bx) \, dx = \frac{1}{b^2 - a^2} \left[ b \sin(ax) \sin(bx) + a \cos(ax) \cos(bx) \right] + C \quad \text{(if } a \neq b) \]

#### Product of Polynomial \(p(x)\) with \(\ln(x), e^x, \cos(x), \sin(x)\)

13. \[ \int x^n \ln(x) \, dx = \frac{1}{n+1} x^{n+1} \ln(x) - \frac{1}{(n+1)^2} x^{n+1} + C \quad \text{(if } n \neq -1) \]
14. \[ \int p(x) e^{ax} \, dx = \frac{1}{a} p(x) e^{ax} - \frac{1}{a} \int p'(x) e^{ax} \, dx \]
   \[= \frac{1}{a} p(x) e^{ax} - \frac{1}{a^2} p'(x) e^{ax} + \frac{1}{a^3} p''(x) e^{ax} - \cdots \quad (+-+-++- \ldots) \]
15. \[ \int p(x) \sin(ax) \, dx = -\frac{1}{a^2} p(x) \cos(ax) + \frac{1}{a} \int p'(x) \cos(ax) \, dx \]
   \[= -\frac{1}{a} p(x) \cos(ax) + \frac{1}{a^2} p'(x) \sin(ax) + \frac{1}{a^3} p''(x) \cos(ax) - \cdots \quad (-+-+-+++) \]
16. \[ \int p(x) \cos(ax) \, dx = \frac{1}{a^2} p(x) \sin(ax) - \frac{1}{a} \int p'(x) \sin(ax) \, dx \]
   \[= \frac{1}{a} p(x) \sin(ax) + \frac{1}{a^2} p'(x) \cos(ax) - \frac{1}{a^3} p''(x) \sin(ax) - \cdots \quad (+-+-+++) \]
**Integer Powers of $\sin(x)$, $\cos(x)$**

17. \[
\int \sin^n(x) \, dx = -\frac{1}{n} \sin^{n-1}(x) \cos(x) + \frac{n-1}{n} \int \sin^{n-2}(x) \, dx + C \quad \text{(if } n > 0)\]

18. \[
\int \cos^n(x) \, dx = -\frac{1}{n} \cos^{n-1}(x) \sin(x) + \frac{n-1}{n} \int \cos^{n-2}(x) \, dx + C \quad \text{(if } n > 0)\]

19. \[
\int \frac{1}{\sin^m(x)} \, dx = \frac{1}{m-1} \frac{\cos(x)}{\sin^{m-1}(x)} + \frac{m-2}{m-1} \int \frac{1}{\sin^{m-2}(x)} \, dx \quad \text{(if } m > 1)\]

20. \[
\int \frac{1}{\sin(x)} \, dx = \frac{1}{2} \ln \left| \frac{\cos(x) - 1}{\cos(x) + 1} \right| + C\]

21. \[
\int \frac{1}{\cos^m(x)} \, dx = \frac{1}{m-1} \frac{\sin(x)}{\cos^{m-1}(x)} + \frac{m-2}{m-1} \int \frac{1}{\cos^{m-2}(x)} \, dx \quad \text{(if } m > 1)\]

22. \[
\int \frac{1}{\cos(x)} \, dx = \frac{1}{2} \ln \left| \frac{\sin(x) + 1}{\cos(x) - 1} \right| + C\]

23. \[
\int \sin^m(x) \cos^n(x) \, dx\]

If $m$ is odd, let $w = \cos(x)$. If $n$ is odd, let $w = \sin(x)$. If both $m$ and $n$ are even and nonnegative, convert all to $\sin(x)$ or all to $\cos(x)$ (using $\cos^2(x) + \sin^2(x) = 1$), and use 17 or 18. If $m$ and $n$ are even and one of them is negative, convert to whichever function is in the denominator and use 19 or 21. If both $m$ and $n$ are even and negative, substitute $w = \tan(x)$.

**Quadratic in the Denominator**

24. \[
\int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \arctan \left( \frac{x}{a} \right) + C\]

25. \[
\int \frac{bx + c}{x^2 + a^2} \, dx = \frac{b}{2} \ln |x^2 + a^2| + \frac{c}{a} \arctan \left( \frac{x}{a} \right) + C \quad \text{(if } a \neq 0)\]

26. \[
\int \frac{1}{(x-a)(x-b)} \, dx = \frac{1}{a-b} \left( \ln |x-a| - \ln |x-b| \right) + C \quad \text{(if } a \neq b)\]

27. \[
\int \frac{cx + d}{(x-a)(x-b)} \, dx = \frac{1}{a-b} \left[ (ac+d) \ln |x-a| - (bc+d) \ln |x-b| \right] + C \quad \text{(if } a \neq b)\]

**Integrands Involving $\sqrt{a^2 + x^2}$, $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$, $a > 0$**

28. \[
\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin \left( \frac{x}{a} \right) + C\]

29. \[
\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left| x + \sqrt{x^2 + a^2} \right| + C\]

30. \[
\int \sqrt{a^2 + x^2} \, dx = \frac{1}{2} \left( x \sqrt{a^2 + x^2} + a^2 \int \frac{1}{\sqrt{a^2 + x^2}} \, dx \right) + C\]

31. \[
\int \sqrt{x^2 - a^2} \, dx = \frac{1}{2} \left( x \sqrt{x^2 - a^2} - a^2 \int \frac{1}{\sqrt{x^2 - a^2}} \, dx \right) + C\]