On a conjecture of Neumann

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Abstract

A conjecture widely attributed to Neumann is that all finite non-desarguesian projective planes contain a Fano subplane. In this note, we show that any finite projective plane of even order which admits an orthogonal polarity contains a Fano subplane. Results of Ganley and Kantor yield that the number of planes of even order \( n \) admitting an orthogonal polarity is not bounded above by any polynomial in \( n \).

1 Introduction

A fundamental question in incidence geometry is about the subplane structure of projective planes. There are relatively few results concerning when a projective plane of order \( k \) is a subplane of a projective plane of order \( n \). Neumann [9] found Fano subplanes in certain Hall planes, which led to the conjecture that every finite non-desarguesian plane contains \( PG(2,2) \) as a subplane (this conjecture is widely attributed to Neumann, though it does not appear in her work).

Johnson [7] and Fisher and Johnson [4] showed the existence of Fano subplanes in many translation planes. Petrak [10] showed that Figueroa planes contain \( PG(2,2) \) and Caliskan and Petrak [3] showed that Figueroa planes of odd order contain \( PG(2,3) \). Caliskan and Moorhouse [2] showed that all Hughes planes contain \( PG(2,2) \) and that the Hughes plane of order \( q^2 \) contains \( PG(2,3) \) if \( q \equiv 5 \) (mod 6). We prove the following.

Theorem 1. Let \( \Pi \) be a finite projective plane of even order which admits an orthogonal polarity. Then \( \Pi \) contains a Fano subplane.

Ganley [5] showed that a finite semifield plane admits an orthogonal polarity if and only if it can be coordinatized by a commutative semifield. A result of Kantor [8] implies that the number of nonisomorphic planes of order \( n \) a power of 2 that can be coordinatized by a commutative semifield is not bounded above by any polynomial in \( n \). Thus, Theorem 1 applies to many projective planes.

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2 Proof of Theorem 1

The proof of Theorem 1 is graph theoretic, and we collect some definitions and results first. Let Π = (P, L, I) be a projective plane of order n. We write p ∈ l or say p is on l if (p, l) ∈ I. Let π be a polarity of Π. That is, π maps points to lines and lines to points, π² is the identity function, and π respects incidence. Then one may construct the polarity graph G_π as follows. V(G_π) = P and p ∼ q if and only if p ∈ π(q). That is, the neighborhood of a vertex p is the line π(p) that p gets mapped to under the polarity. If p ∈ π(p), then p is an absolute point and the vertex p will have a loop on it. A polarity is orthogonal if exactly n + 1 points are absolute. We note that as neighborhoods in the graph represent lines in the geometry, each vertex in G_π has exactly n + 1 neighbors (if v is an absolute point, it has exactly n neighbors other than itself). We provide proofs of the following preliminary observations for completeness.

Lemma 1. Let Π be a projective plane with polarity π, and G_π be the associated polarity graph.

(a) For all u, v ∈ V(G_π), u and v have exactly 1 common neighbor.
(b) G_π is C_4 free.
(c) If u and v are two absolute points of G_π, then u ∼ v.
(d) If v ∈ V(G_π), then the neighborhood of v induces a graph of maximum degree at most 1.
(e) Let e = uv be an edge of G_π such that neither u nor v is an absolute point. Then e lies in a unique triangle in G_π.

Proof. To prove (a), let u and v be an arbitrary pair of vertices in V(G_π). Because Π is a projective plane, π(u) and π(v) meet in a unique point. This point is the unique vertex in the intersection of the neighborhood of u and the neighborhood of v. (b) and (c) follow from (a).

To prove (d), if there is a vertex of degree at least 2 in the graph induced by the neighborhood of v, then G_π contains a 4-cycle, a contradiction by (b).

Finally, let u ∼ v and neither u nor v an absolute point. Then by (a) there is a unique vertex w adjacent to both u and v. Now uwv is the purported triangle, proving (e).

Proof of Theorem 1. We will now assume Π is a projective plane of even order n, that π is an orthogonal polarity, and that G_π is the corresponding polarity graph (including loops). Since n is even and π is orthogonal, a classical theorem of Baer ([1], see also Theorem 12.6 in [6]) says that the n + 1 absolute points under π all lie on one line. Let a_1, ..., a_{n+1} be the set of absolute points and let l be the line containing them. Then there is some p ∈ P such that π(l) = p. This means that in G_π, the neighborhood of p is exactly the set of points \{a_1, ..., a_{n+1}\}. For 1 ≤ i ≤ n + 1, let N_i be the neighborhood of a_i. Then by Lemma 1.b, N_i ∩ N_j = ∅ if i ≠ j. Further, counting gives that

\[ V(G_π) = p \cup \left( \bigcup_{i=1}^{n+1} a_i \right) \cup \left( \bigcup_{i=1}^{n+1} N_i \right). \]
Figure 1: $ER_2^o$

Let $ER_2^o$ be the graph on 7 points which is the polarity graph (with loops) of $PG(2, 2)$ under the orthogonal polarity.

**Lemma 2.** If $ER_2^o$ is a subgraph of $G_π^o$, then $Π$ contains a Fano subplane.

**Proof.** Let $v_1, \ldots, v_7$ be the vertices of a subgraph $ER_2^o$ of $G_π^o$. Let $l_i = \pi(v_i)$ for $1 \leq i \leq 7$. Then the lines $l_1, \ldots, l_7$ in $Π$ restricted to the points $v_1, \ldots, v_7$ form a point-line incidence structure, and one can check directly that it satisfies the axioms of a projective plane.

Thus, it suffices to find $ER_2^o$ in $G_π^o$. To find $ER_2^o$ it suffices to find distinct $i, j, k$ such that there are $v_i \in N_i$, $v_j \in N_j$, and $v_k \in N_k$ where $v_i v_j v_k$ forms a triangle in $G_π^o$, for then the points $p, a_i, a_j, a_k, v_i, v_j, v_k$ yield the subgraph $ER_2^o$. Now note that for all $i$, and for $v \in N_i$, $v$ has exactly $n$ neighbors that are not absolute points. There are $n+1$ choices for $i$ and $n-1$ choices for $v \in N_i$. As each edge is counted twice, this yields

$$\frac{n(n-1)(n+1)}{2}$$

edges with neither end an absolute point. By Lemma 1.e, there are at least

$$\frac{n^3 - n}{6}$$

triangles in $G_π^o$. By Lemma 1.c, there are no triangles incident with $p$, by Lemma 1.b, there are no triangles that have more than one vertex in $N_i$ for any $i$, and by Lemma 1.d there are at most $\left\lfloor \frac{n-1}{2} \right\rfloor = \frac{n}{2} - 1$ triangles incident with $a_i$ for each $i$. Therefore, by (1), there are at least

$$\frac{n^3 - n}{6} - (n+1)\left(\frac{n}{2} - 1\right)$$

copies of $ER_2^o$ in $G_π^o$. This expression is positive for all even natural numbers $n$. $\square$
3 Concluding Remarks

First, we note that the proof of Theorem 1 actually implies that there are $\Omega(n^3)$ copies of $PG(2,2)$ in any plane satisfying the hypotheses, and echoing Petrak [10], perhaps one could find subplanes of order 4 for $n$ large enough. We also note that it is crucial in the proof that the absolute points form a line. When $n$ is odd, the proof fails (as it must, since it is unspecified if $\Pi$ is desarguesian or not).

Acknowledgments

The author would like to thank Gary Ebert and Eric Moorhouse for helpful comments.

References


