Increasing paths in edge-ordered graphs

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A game

Let’s play a game

What is the longest increasing path you can find?
Definitions

**Definition**

An *edge-ordering* $\phi$ of a graph $G$ is a bijection $\phi : E(G) \rightarrow \{1, \ldots, |E(G)|\}$.

**Definition**

Given an edge-ordering $\phi$, and *increasing path* is a path $e_1 e_2 \cdots e_k$ such that $\phi(e_1) < \phi(e_2) < \cdots < \phi(e_k)$.

Note that a path is a self-avoiding walk, ie no vertex is visited more than once.
A game

There is an increasing path of length at least 4.
Our goal is to find a long increasing path. Our opponent’s goal is to order the edges so that we cannot find a long increasing path.
Our opponent

Our goal is to find a long increasing path. Our opponent’s goal is to order the edges so that we cannot find a long increasing path.
Max-min problem

If both players play optimally, how long will the longest increasing path be? Given a graph $G$, define $f(G)$ to be this length.

Definition

Fix a graph $G$. Define

$$f(G) = \min_{\phi} \text{length of longest increasing path under } \phi$$

where $\phi$ runs through all edge-orderings.
Chvátal and Komlós ask about $f(K_n)$ in 1971.

Graham and Kleitman show $f(K_n) \geq \sqrt{n-1}$ in 1973.

Rödl shows if $G$ has average degree $d$, then $f(G) \gtrsim \sqrt{d}$ in 1973.

A series of upper bounds for $f(K_n)$ follow, settling on $f(K_n) < (1/2 + o(1))n$ by Calderbank, Chung, and Sturtevant in 1984.

Theorem (GRWC 2014)

Let $Q_d$ denote the $d$-dimensional hypercube. Then for all $d \geq 2$,

$$f(Q_d) \geq \frac{d}{\log d}.$$ 

Theorem (GRWC 2014)

Let $\omega$ be any function tending to infinity, and $p \leq \frac{\log n}{\sqrt{n}} \omega(n)$. Then with probability tending to 1,

$$f(G(n, p)) \geq \frac{(1 - o(1))np}{\omega(n) \log n}.$$ 

Both of these bounds are tight up to the logarithmic factor.
Progress on $f(K_n)$

Our theorem shows that a random graph with expected degree just slightly larger than $\sqrt{n}$ satisfies the same lower bound that Graham and Kleitman showed for $K_n$. We thought that this was good evidence that the lower bound for $f(K_n)$ was not correct.

**Theorem (Milans)**

\[ f(G) = \Omega \left( \left( \frac{n}{\log n} \right)^{2/3} \right) . \]
The pedestrian argument

**Theorem (Graham-Kleitman 1973, Rödl 1973)**

*Every edge-ordering of $K_n$ contains an increasing path of length at least $\sqrt{n - 1}$. That is*

$$f(K_n) \geq \sqrt{n - 1}.$$ 

Place a pedestrian on each vertex.
The pedestrian argument

Call out the edges in order. The two pedestrians switch places unless it would cause one of them to revisit a vertex she has already seen.
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The pedestrian argument
The blue pedestrian has walked an increasing path of length 4 (1 − 4 − 10 − 15).
The pedestrian argument

Theorem (Graham-Kleitman 1973, Rödl 1973)
Every edge-ordering of $K_n$ contains an increasing path of length at least $\sqrt{n-1}$. That is

$$f(K_n) \geq \sqrt{n-1}.$$  

Proof:
- Suppose each pedestrian walks $\leq k$ steps during this process.
- Then at most $\frac{kn}{2}$ edges are traversed.
- Each pedestrian declines to walk an edge at most $\binom{k+1}{2} - k$ times.

$$\text{edges walked} + \text{edges declined} = \binom{n}{2} \leq \frac{kn}{2} + \binom{k}{2} n = \frac{k^2 n}{2}.$$
The pedestrian argument

Consider the pedestrian algorithm on an arbitrary graph $G$. Every edge in $G$ is either traversed or is declined by some pedestrian. An edge may only be declined if it is contained in the subgraph induced by the path walked by a pedestrian.

**Lemma**

Let $G$ be any graph. If $f(G) < k$, there exist sets $V_1, \ldots, V_n \subset V(G)$ such that $|V_i| \leq k$ and every edge of $G$ is contained in a subgraph induced by some $V_j$.

In particular,

$$n \cdot (\# \text{ edges in densest subgraph on } f(G) \text{ vertices}) \geq |E(G)|.$$
The hypercube

\[ n \cdot (\# \text{ edges in densest subgraph on } f(G) \text{ vertices}) \geq |E(G)| \]

**Theorem (GRWC 2014)**

\[ f(Q_d) \geq \frac{d}{\log d} \]

*Proof: Lemma:* Any subgraph of a hypercube has density less than or equal to a subhypercube of the same size.
The random graph

**Theorem (GRWC 2014)**

Let $\omega(n)$ be a function tending to infinity arbitrarily slowly. Then for any $p \geq \frac{\log n}{\sqrt{n}} \omega(n)$, with probability tending to 1

$$f(G(n,p)) \geq \frac{(1 - o(1))np}{\omega(n) \log n}$$

**Proof:** The graphs induced by the pedestrians’ paths must cover all of the edges of $G(n,p)$. If $f(G(n,p)) \leq \frac{np}{\omega(n) \log n}$, we get a lower bound on the number of pairs that cannot be edges. The probability that this occurs is $o\left((f(G(n,p))^n\right)^n)$, i.e. it is so unlikely that even adding up over all possible paths for the pedestrians the probability that it occurs is still $o(1)$. 
Upper Bounds

Our opponent wants to label the edges of $G$ so that there is no long increasing path. Constructing an edge-labeling yields an upper bound on $f(G)$.

A first strategy: Consider a proper edge-coloring of a graph $G$ with colors $c_1, \cdots, c_k$. Label the edges with color $c_1$ with the smallest labels. Label the edges with color $c_2$ with the next smallest labels. Continue this process. Any increasing path can use at most one edge of each color.
Open problems

Lavrov and Loh studied a variant of this problem. What happens when the edges of $K_n$ are ordered randomly?

**Theorem (Lavrov-Loh)**

With probability tending to 1, a random edge-ordering of $K_n$ has a monotone path of length at least $0.85n$. With probability at least $1/e - o(1)$, a random edge-ordering of $K_n$ has an increasing Hamiltonian path.

**Conjecture**

With probability tending to 1, a random edge-ordering of $K_n$ contains an increasing Hamiltonian path.
Open problems

- Improve the lower bound $f(K_n) = \Omega \left( \frac{n}{\log n}^{2/3} \right)$.
- Does $f(Q_d) = d$?
- Are there graphs $G$ with $\Delta(G) = k$ and $f(G) = k + 1$?
- Show a random edge-ordering of $K_n$ contains an increasing Hamiltonian path with probability tending to 1.