Gaps of saddle connection directions for some branched covers of tori

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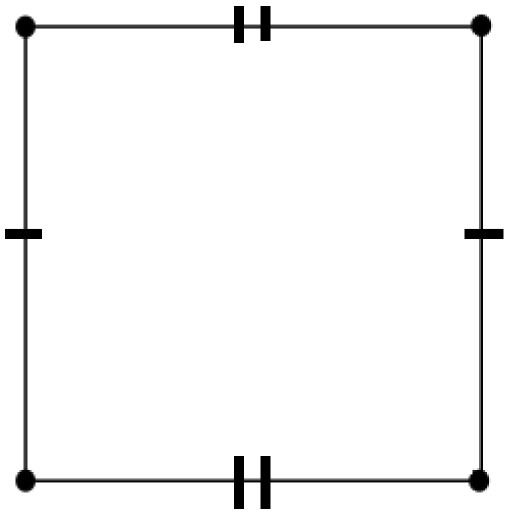
Translation surfaces

A **translation surface** is a collection of polygons with edge identifications given by translations.



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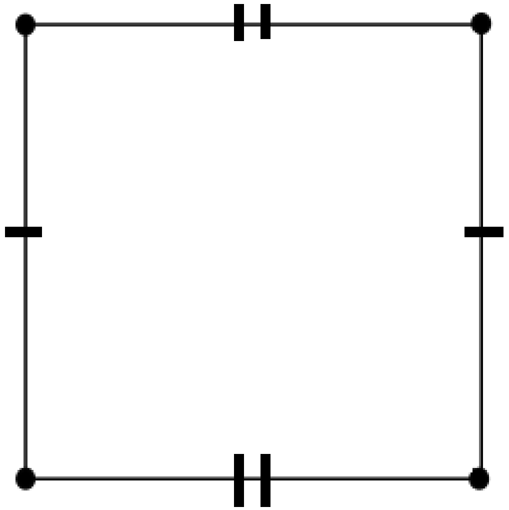

Torus

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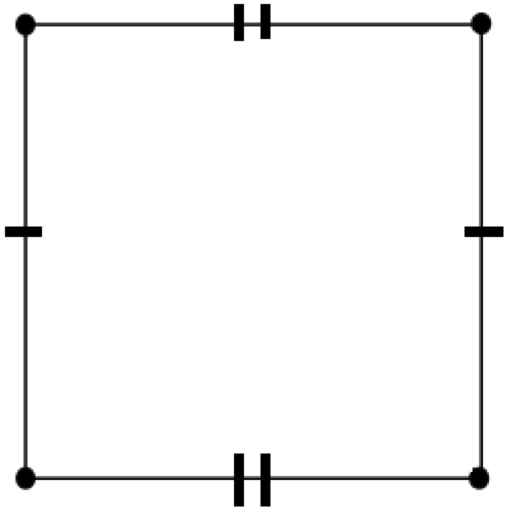

Torus

- Genus 1

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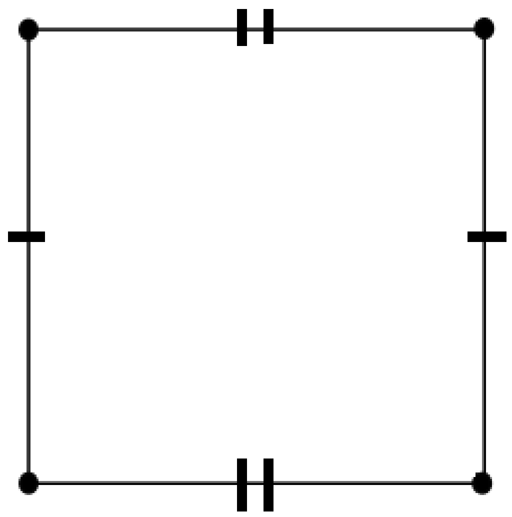
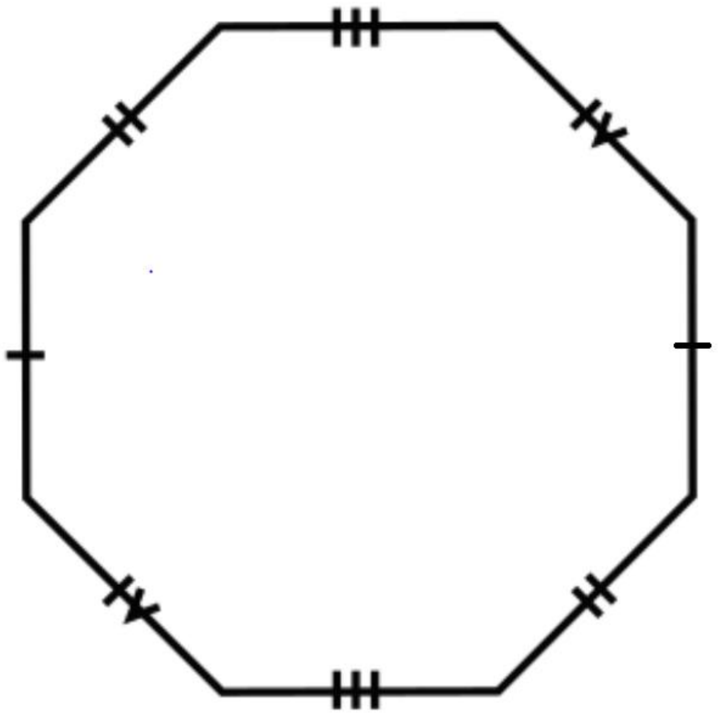


Torus

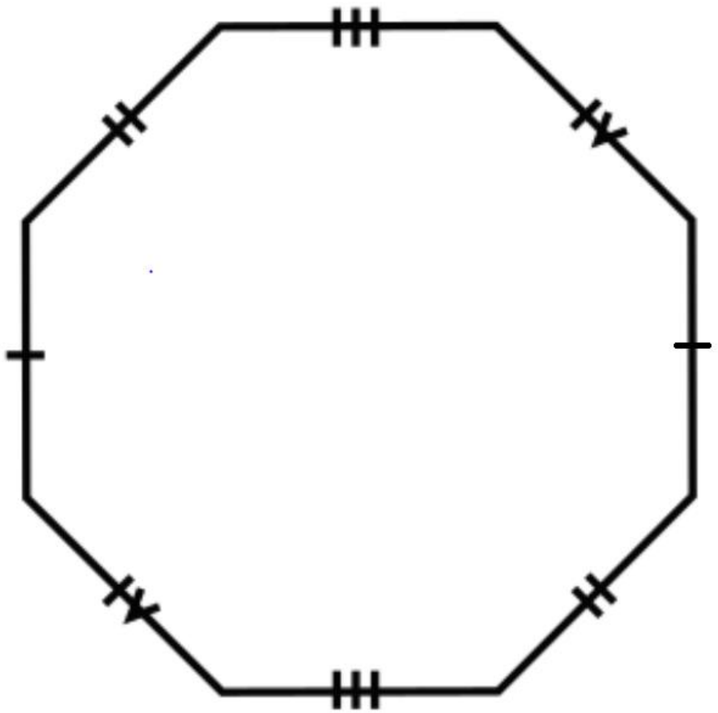
- Genus 1
- Flat geometry everywhere.

Octagon



Regular Octagon:

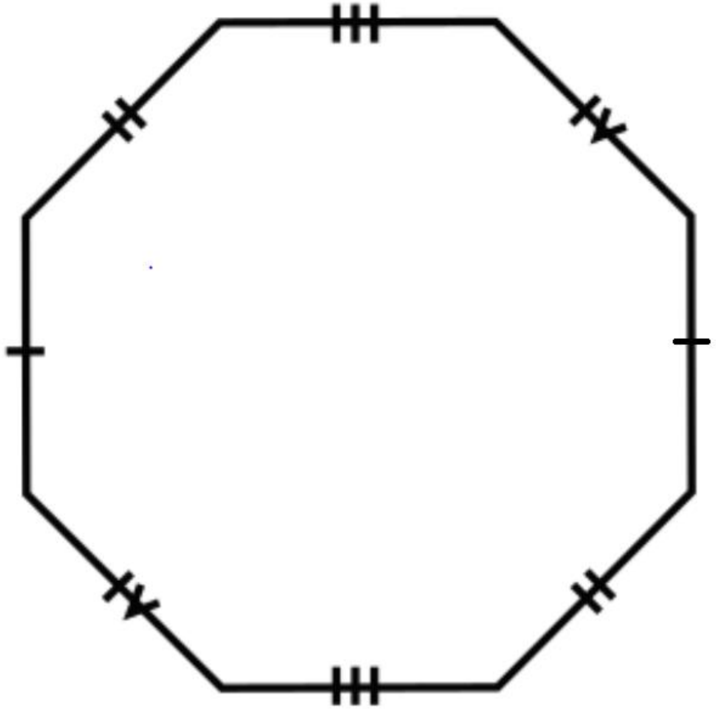
Octagon



Regular Octagon:

- Genus 2

Octagon

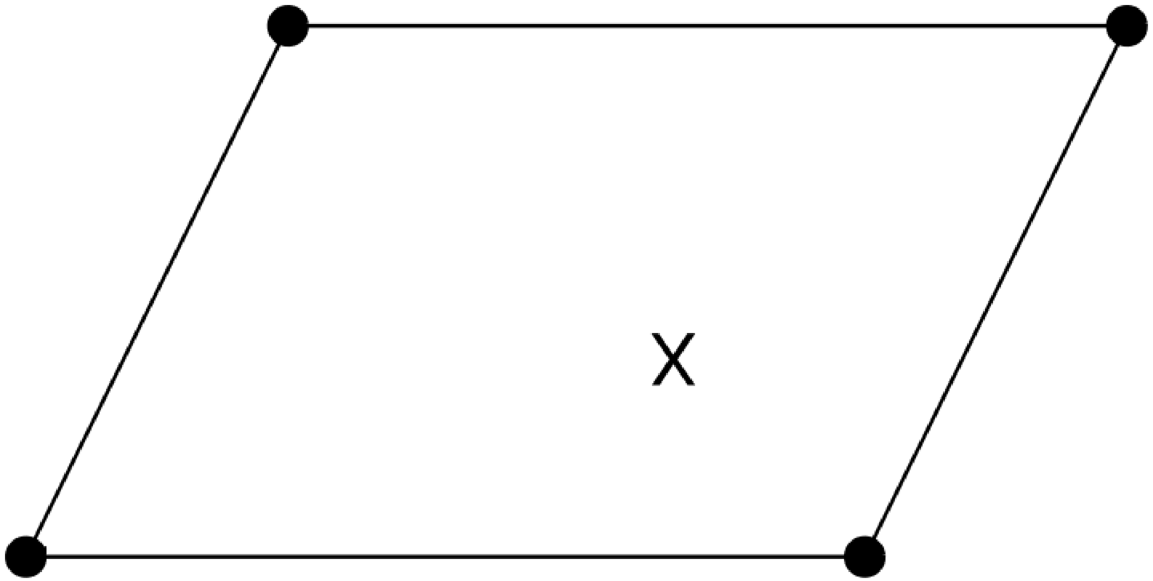


Regular Octagon:

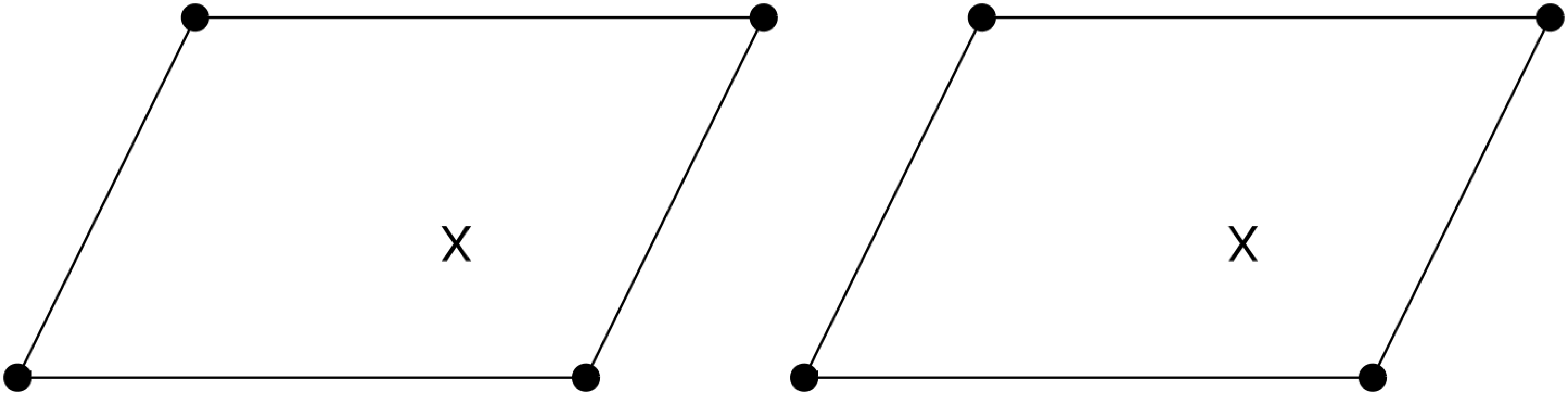
- Genus 2
- Single cone point of angle 6π

Doubled slit torus construction

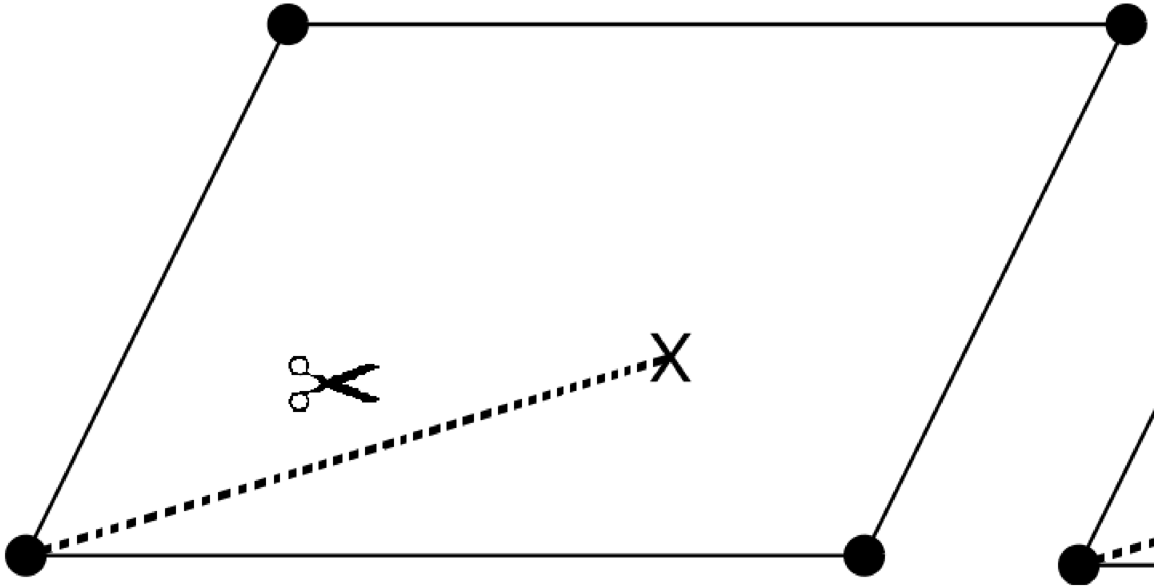
Take a flat torus and mark two points

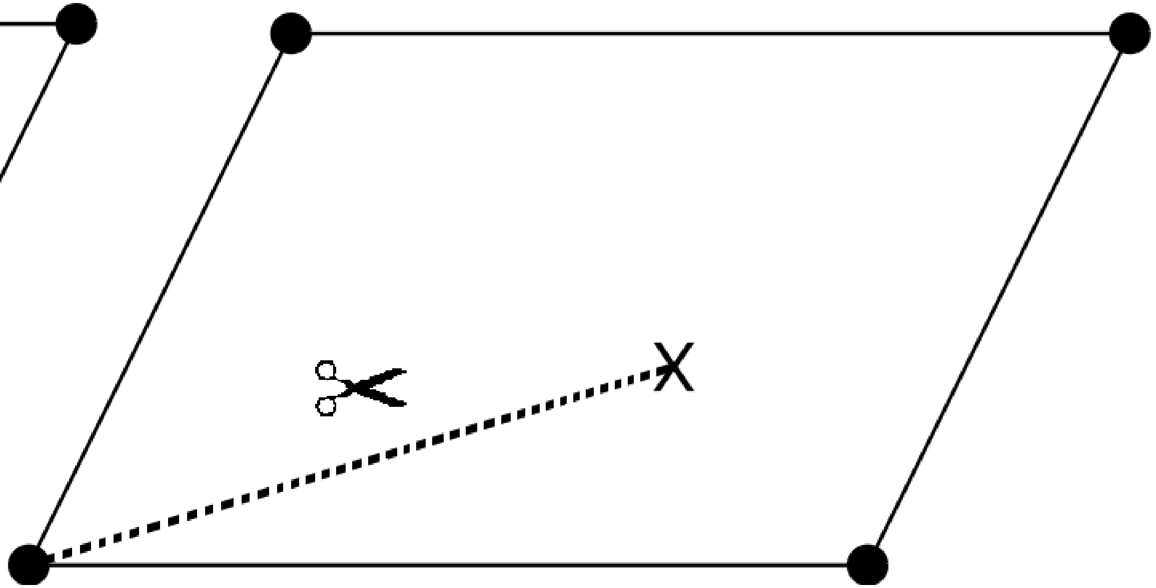


Take an identical copy of the twice-marked torus

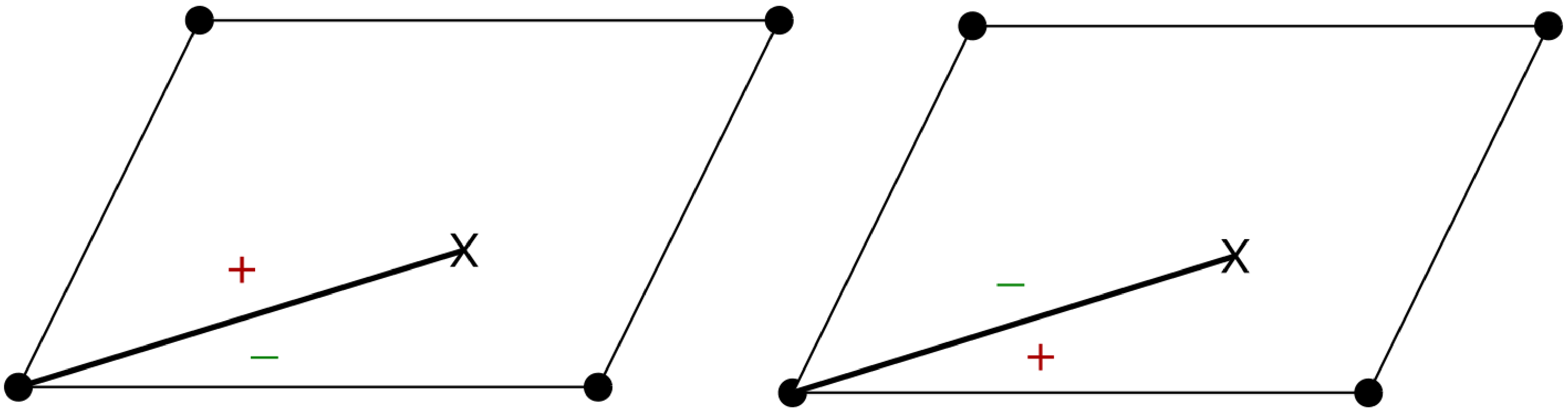


Cut a slit between the marked points

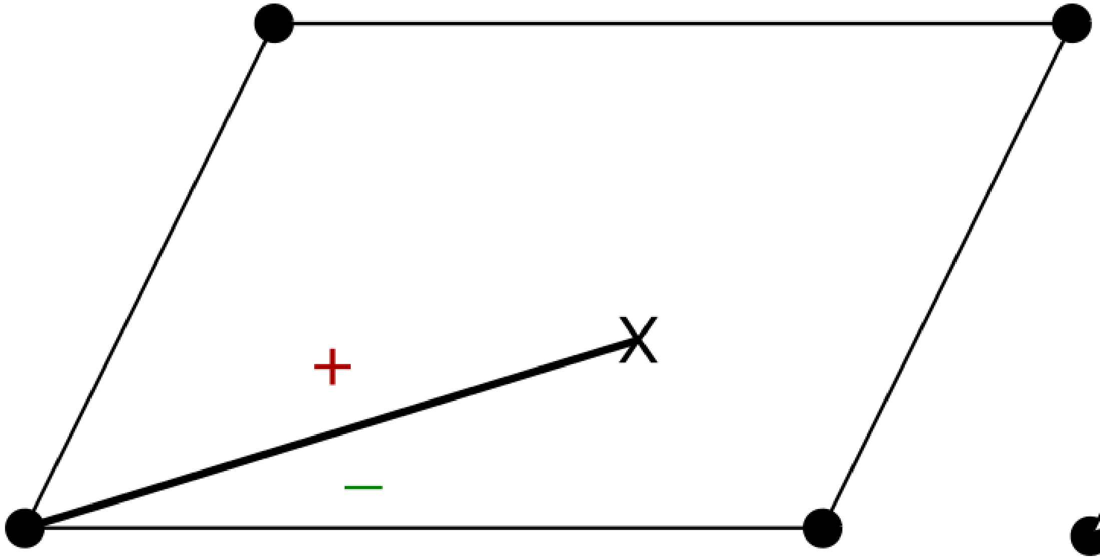


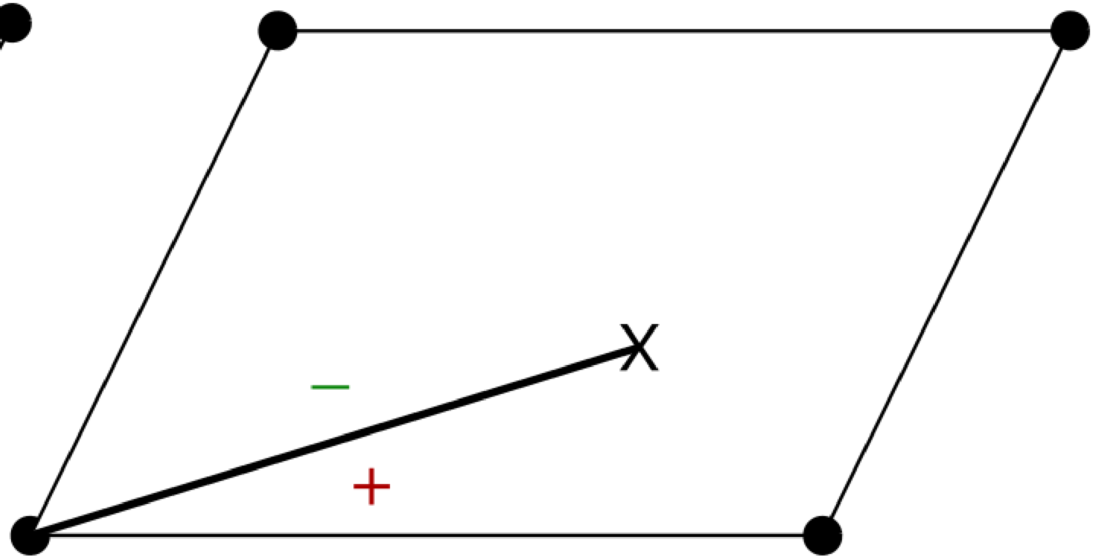


Glue opposite sides of the slit together



Doubled Slit Torus





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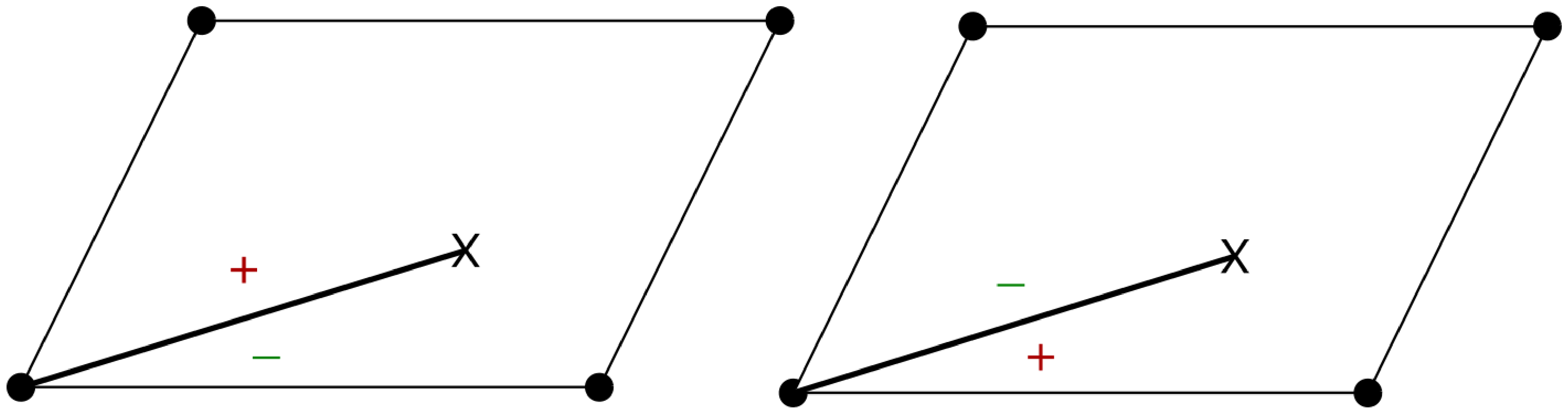
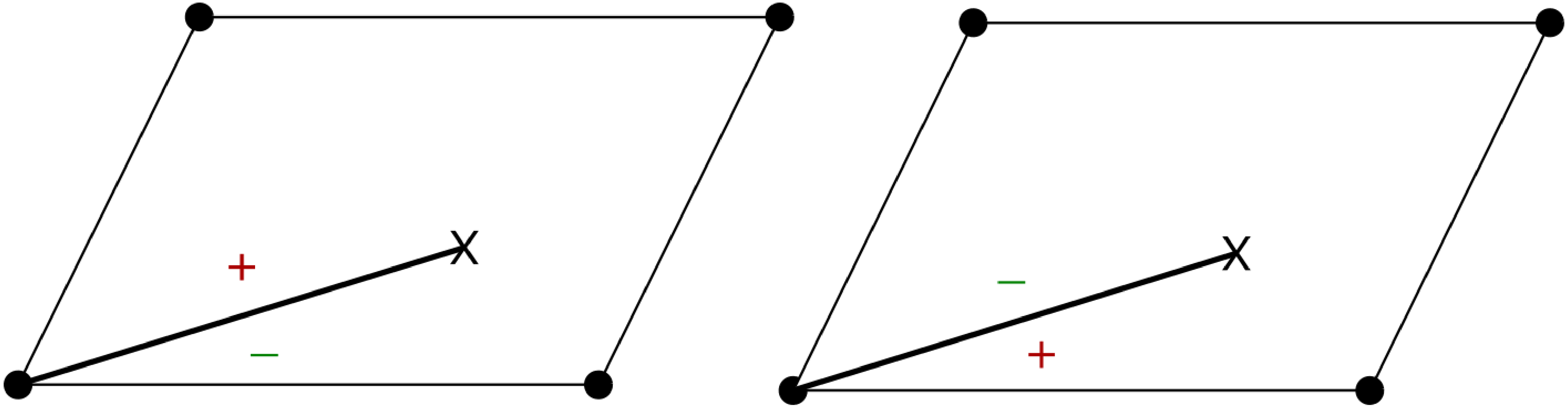


Genus 2 surface

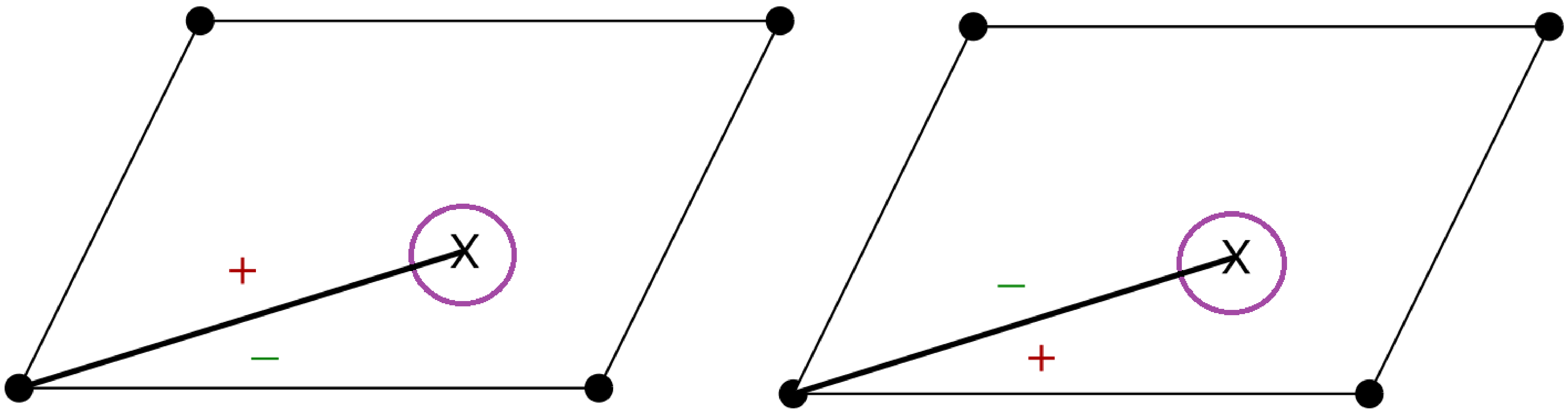
Doubled Slit Torus



Genus 2 surface

2 cone type singularities of angle 4π

Doubled Slit Torus



Genus 2 surface

2 cone type singularities of angle 4π

Why doubled slit tori?

(Topology)

Are a natural construction of a higher genus surface from genus 1 surfaces.

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(Dynamics)

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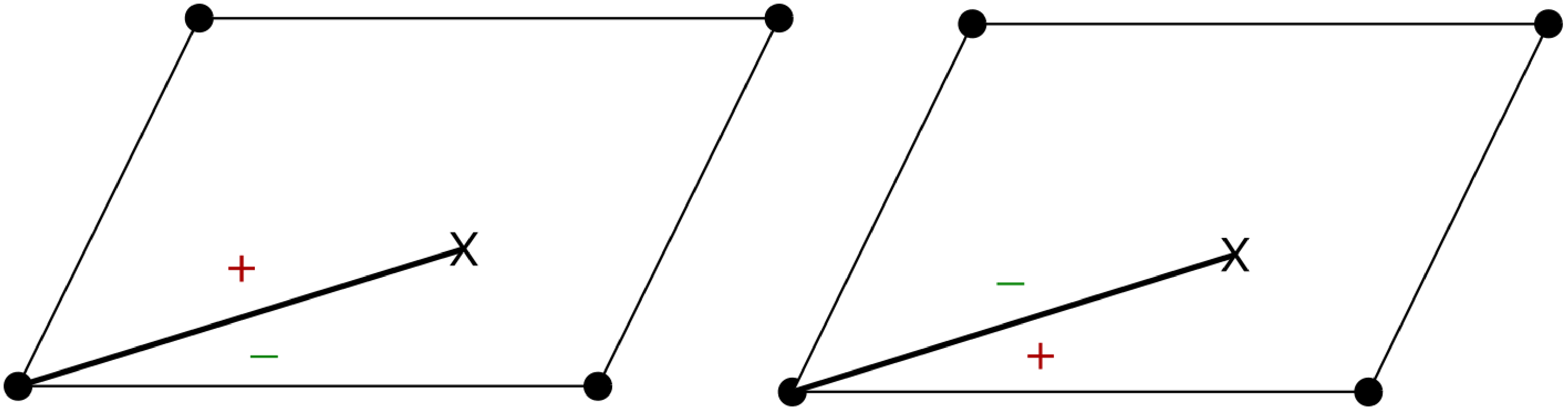
(Geometry)

Are examples of translation surfaces.

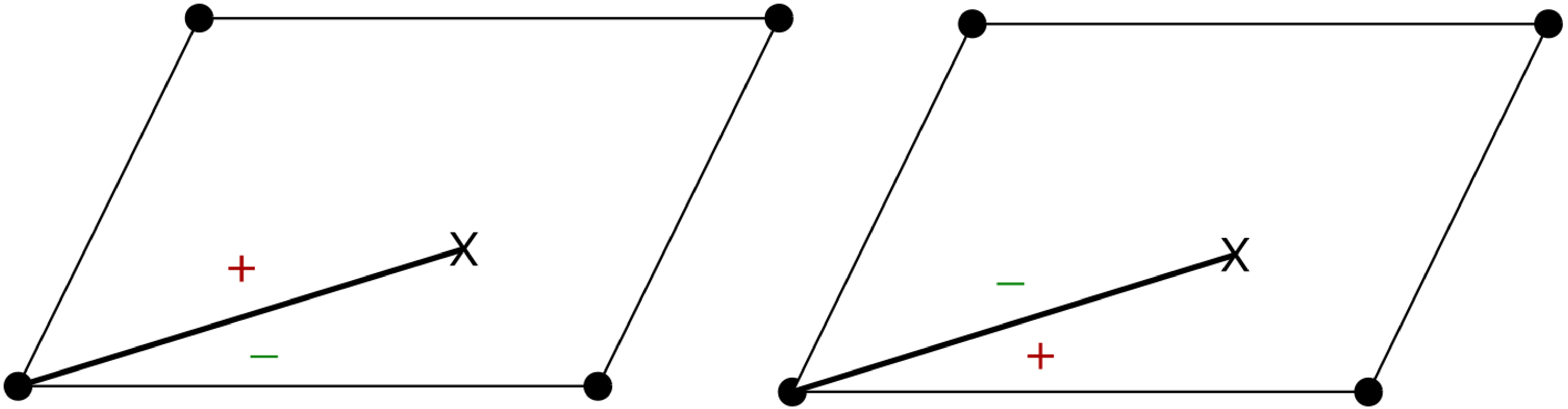
Translation structure

Embedding into complex plane endows the surface with a Riemann surface structure X



Translation structure

Embedding into complex plane endows the surface with a Riemann surface structure X and the holomorphic differential dz.



Translation surfaces

More generally any pair (X, ω) where X is a Riemann surface and ω is a non-zero holomorphic differential is called a ***translation surface***.

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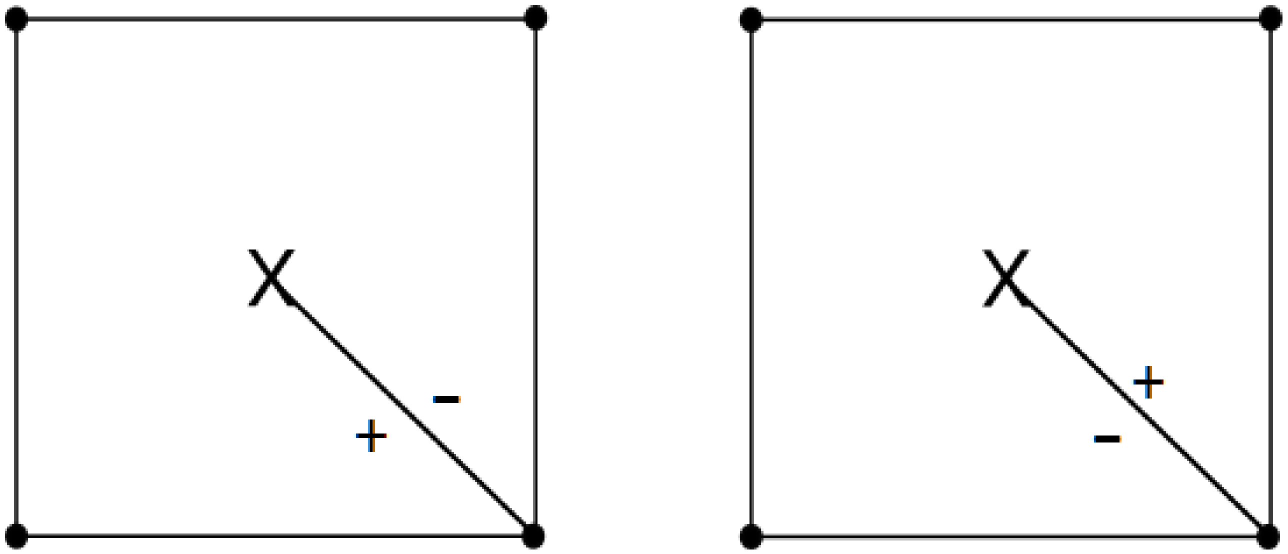
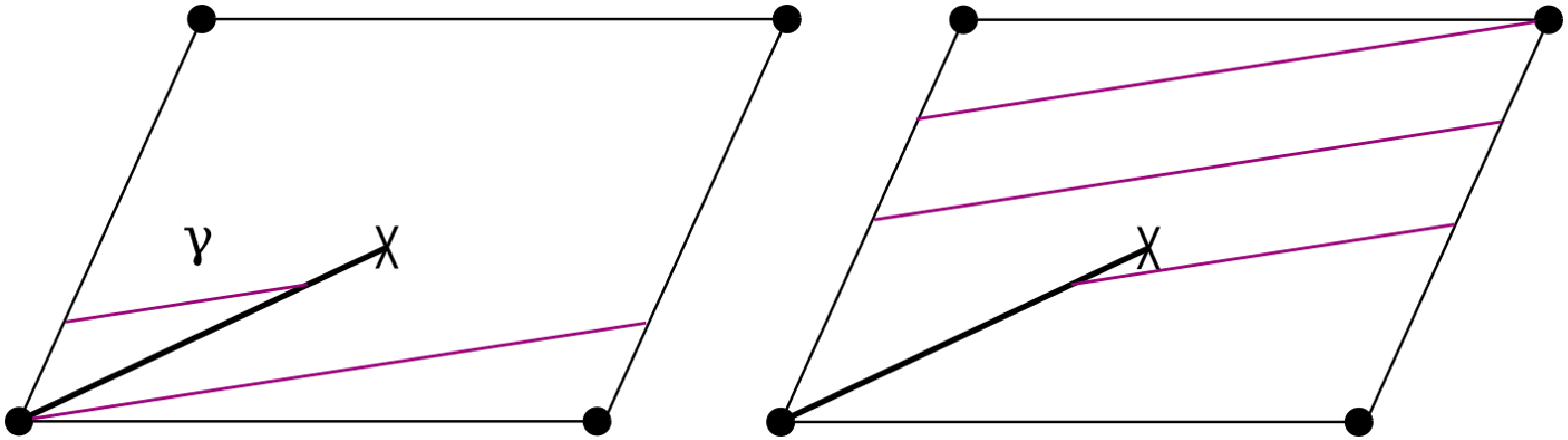
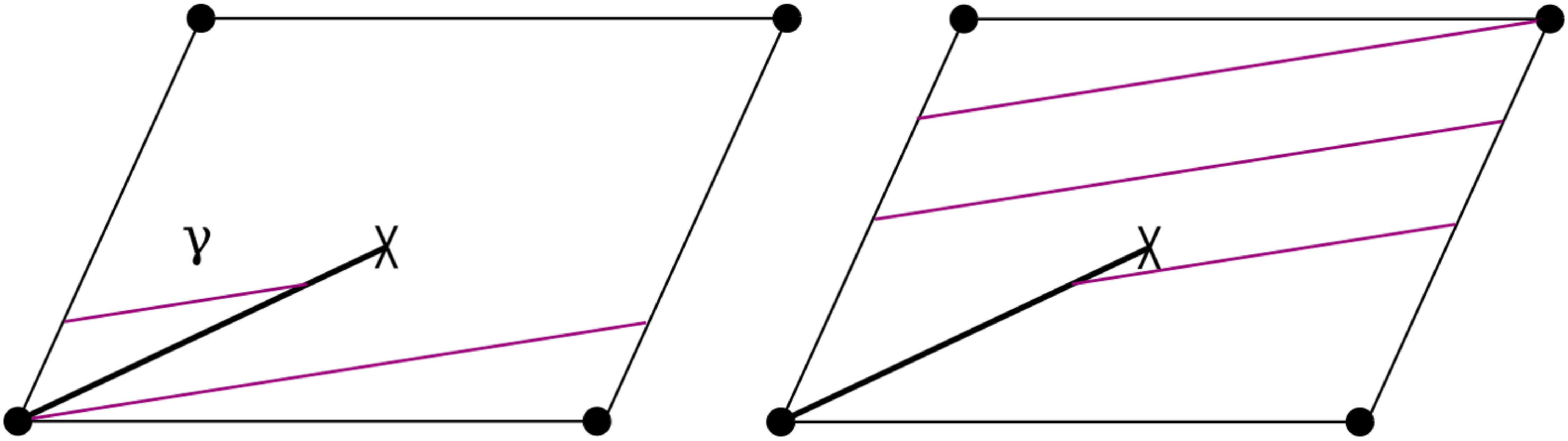
The holomorphic differential allows us to measure lengths and gives a sense of direction.

We are interested in paths on doubled slit tori

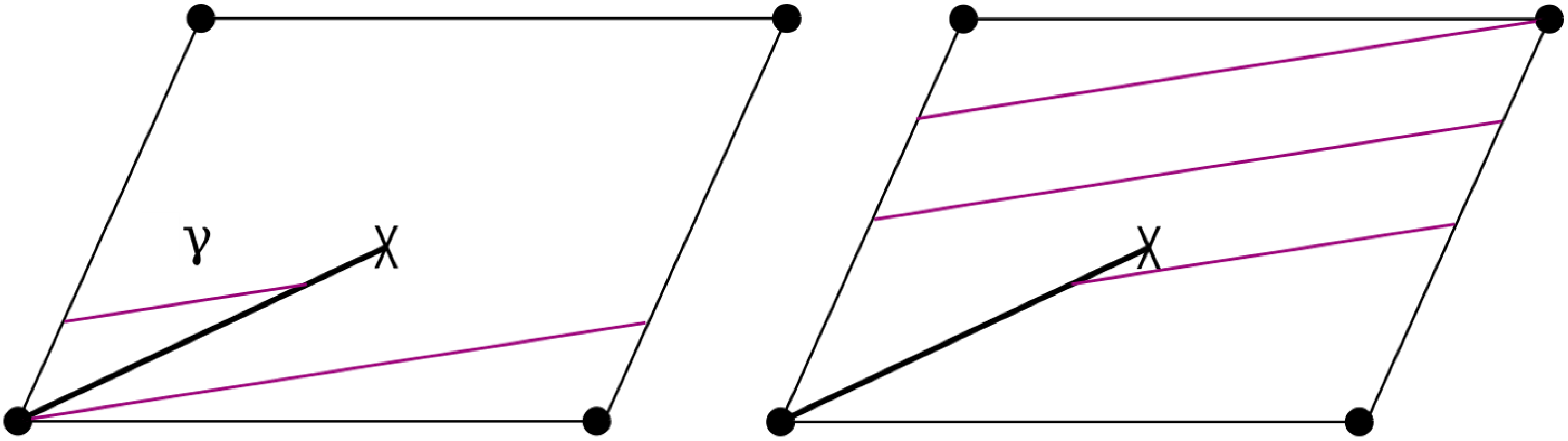


A ***saddle connection*** is a straight-line trajectory starting and ending at a cone type singularity.





Associated to each saddle connection is the ***holonomy vector***.



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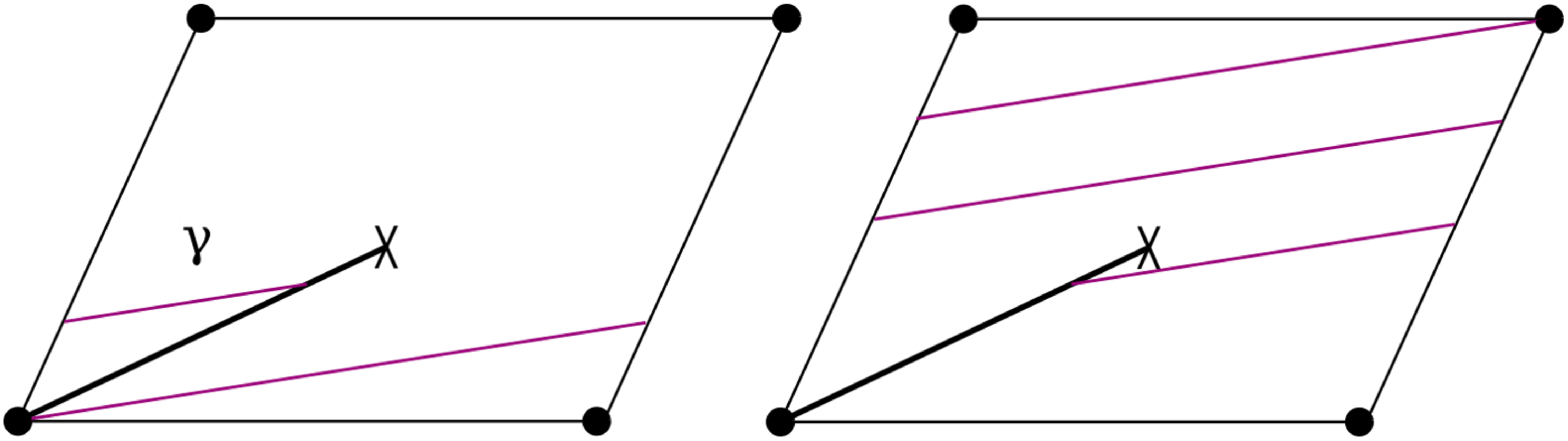
$$\int_\gamma dz = 4 + i$$



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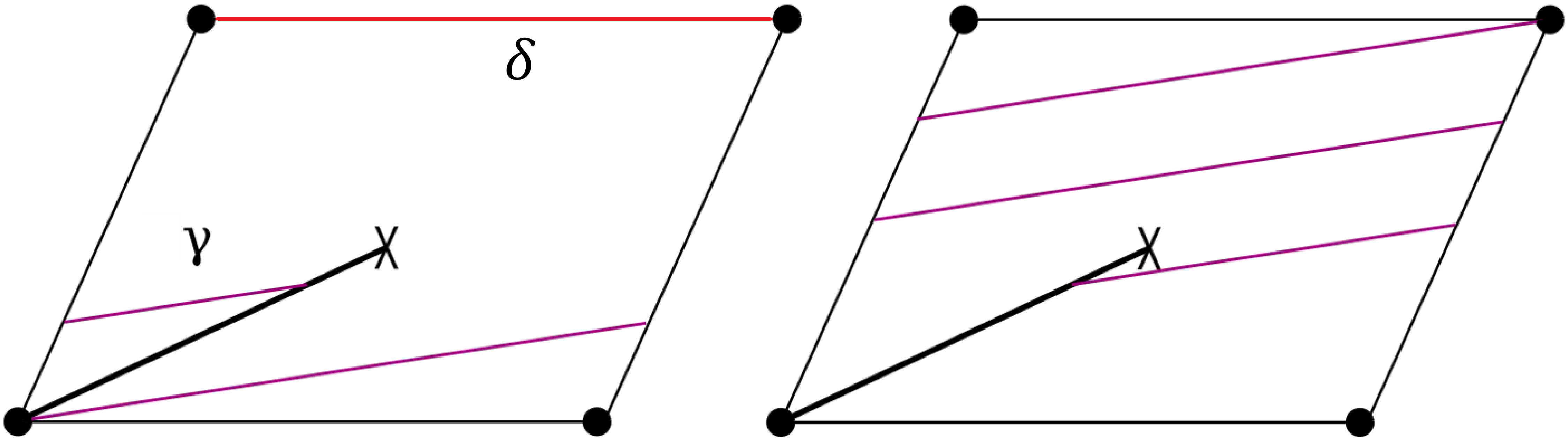
$$\int_\gamma \; dz = 4 + i \text{ or } \binom{4}{1}$$


δ
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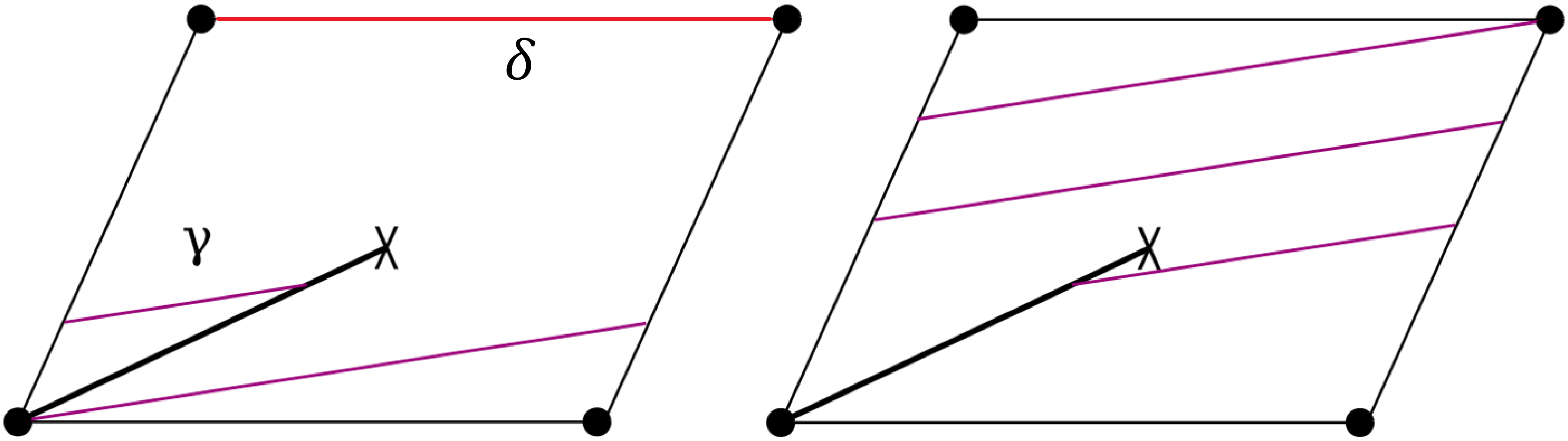
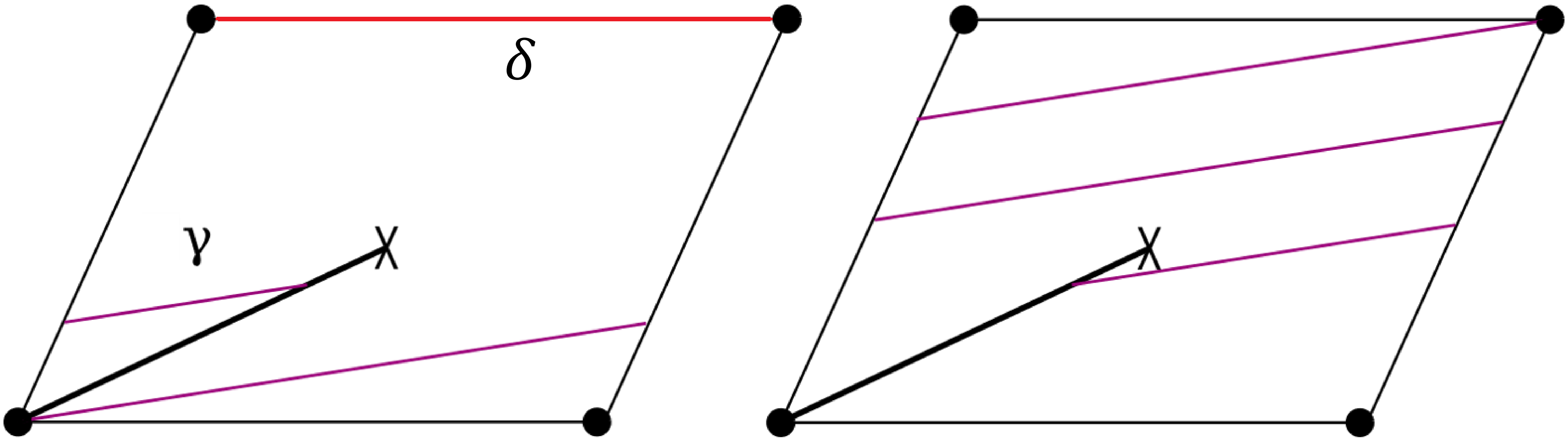


$$\int_\delta dz = 1 + 0i \text{ or } \begin{pmatrix}1\\0\end{pmatrix}$$



$$\int_{\gamma} dz = 4 + i \text{ or } \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad \text{and} \quad \int_{\delta} dz = 1 + 0i \text{ or } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

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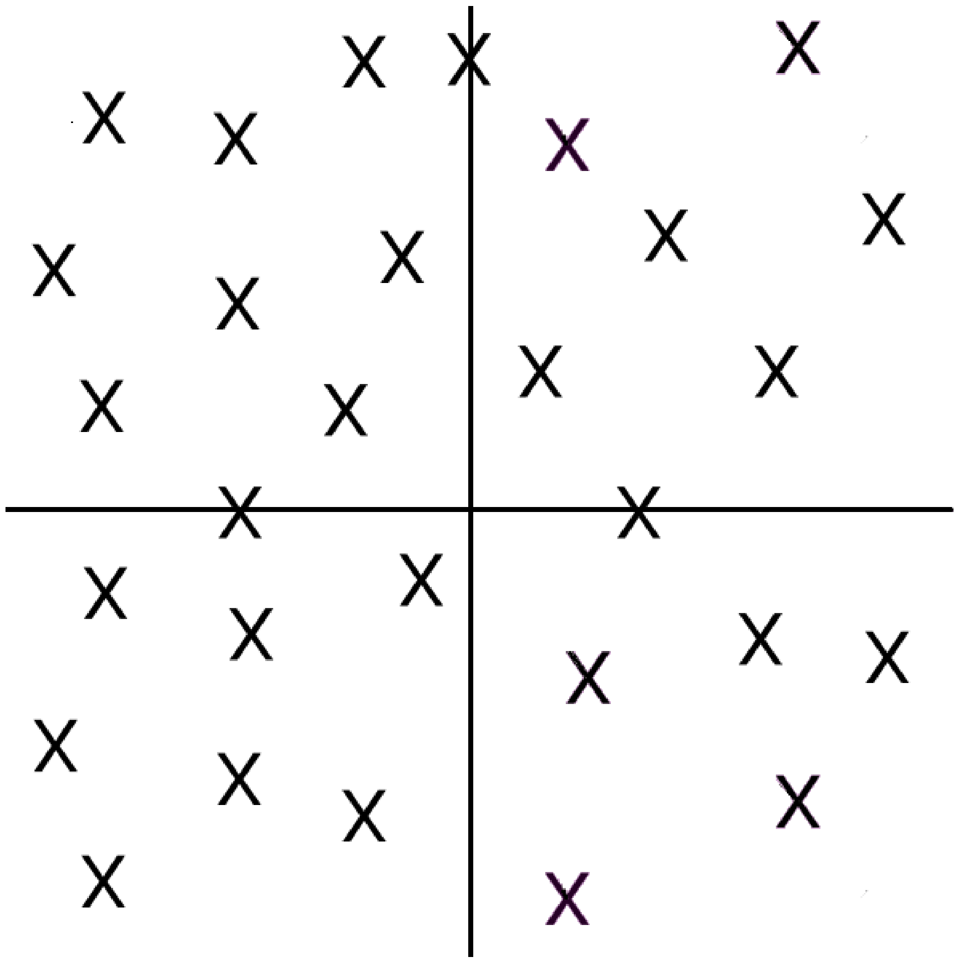
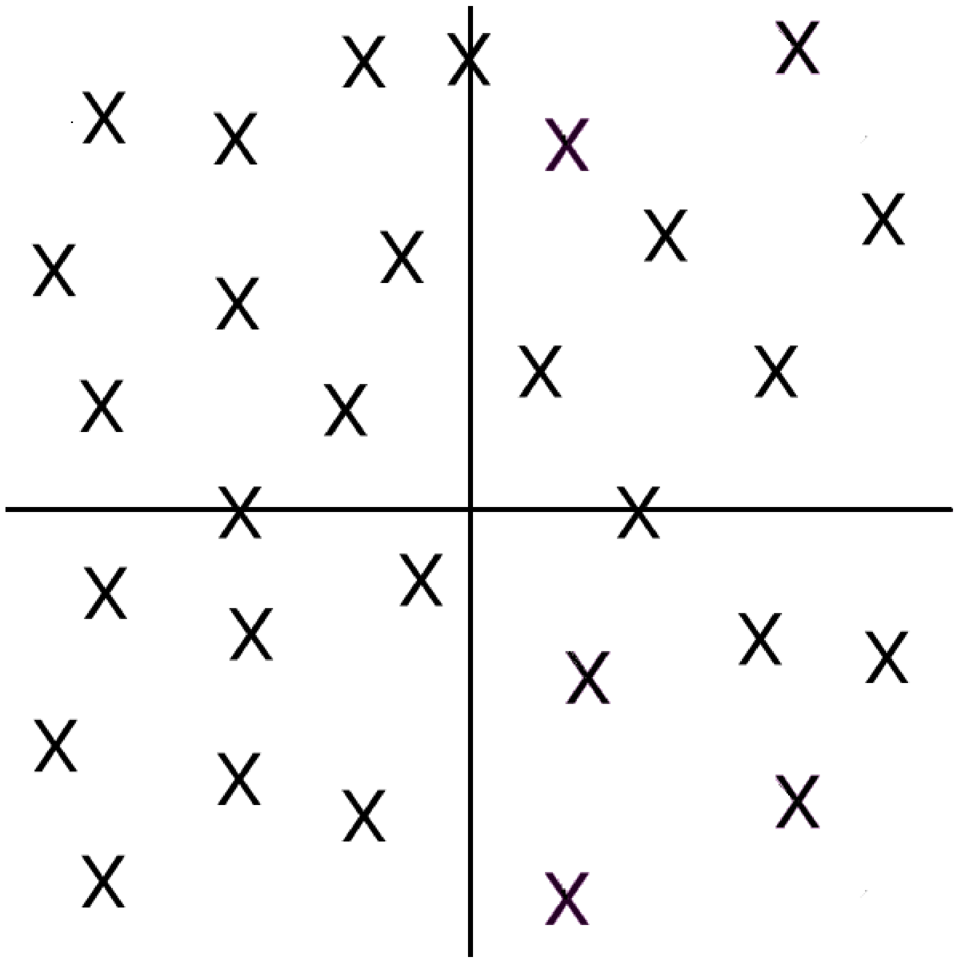


Discreteness

Λ_ω

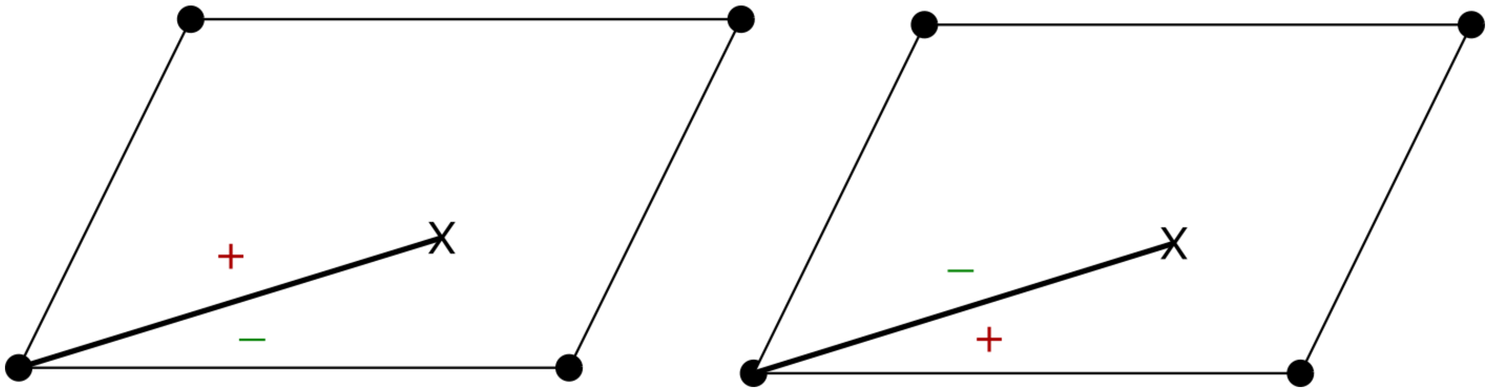


Let Λ_ω denote the set of all holonomy vectors.

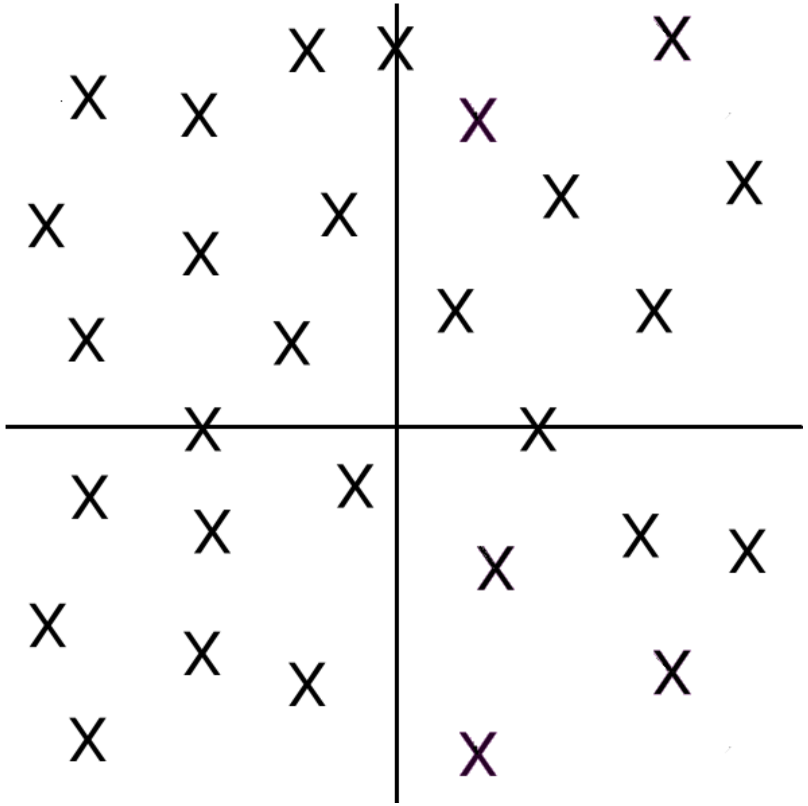
Veech: Λ_ω is a discrete subset!

How random are the holonomy vectors?

(X, ω)



Λ_ω



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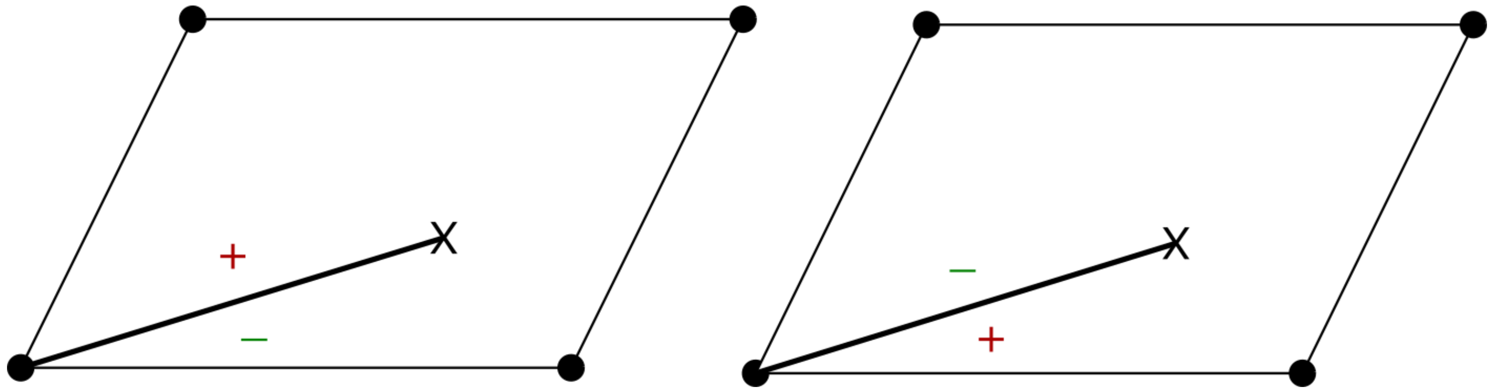

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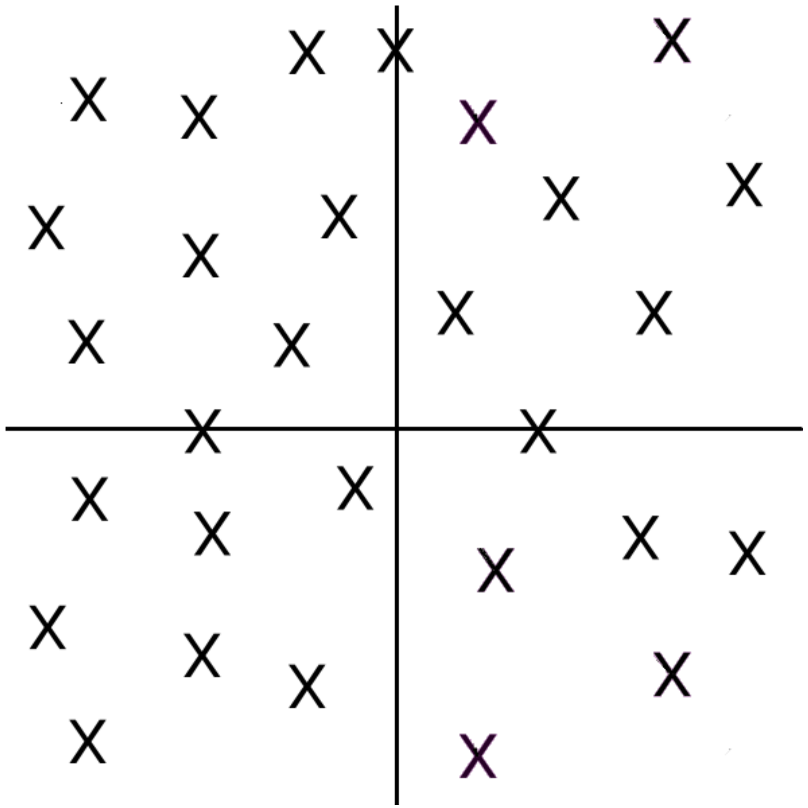
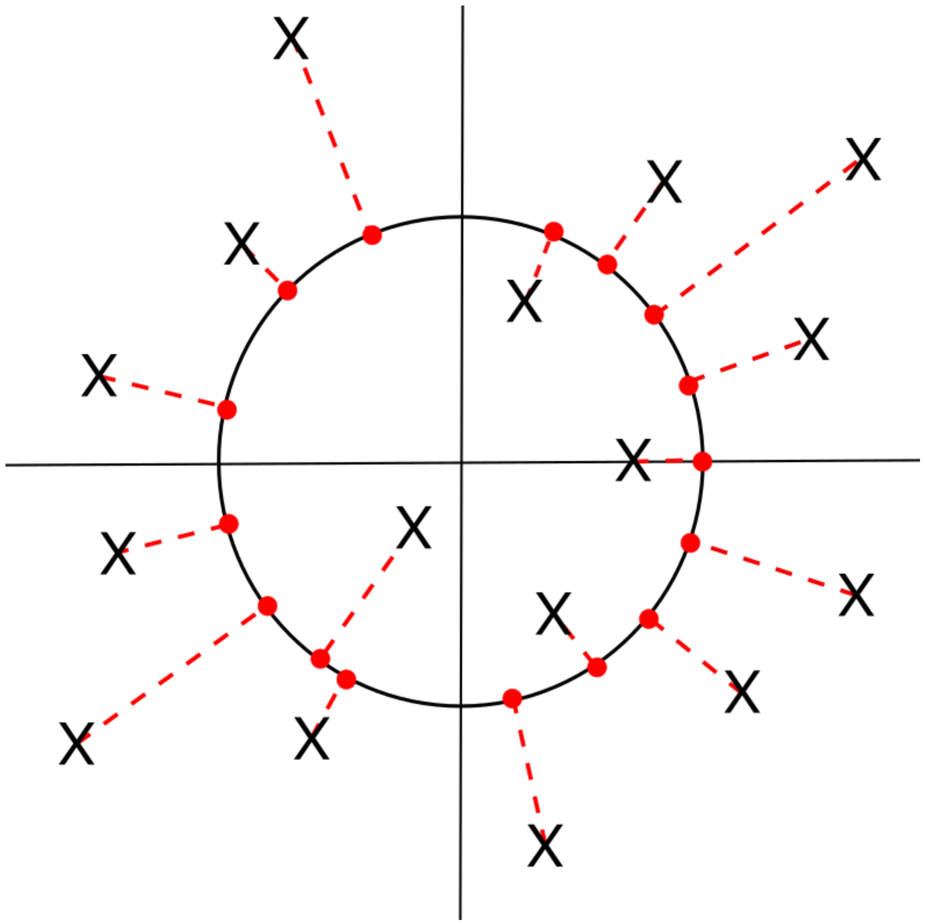


Angles as a test of randomness



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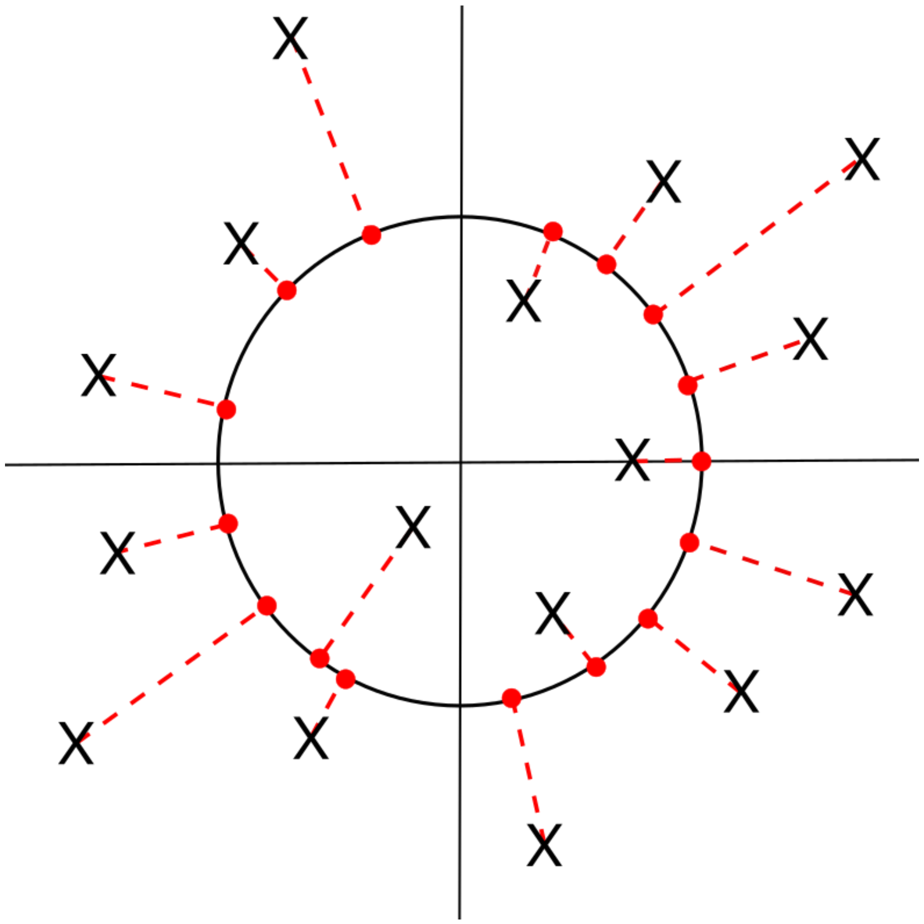
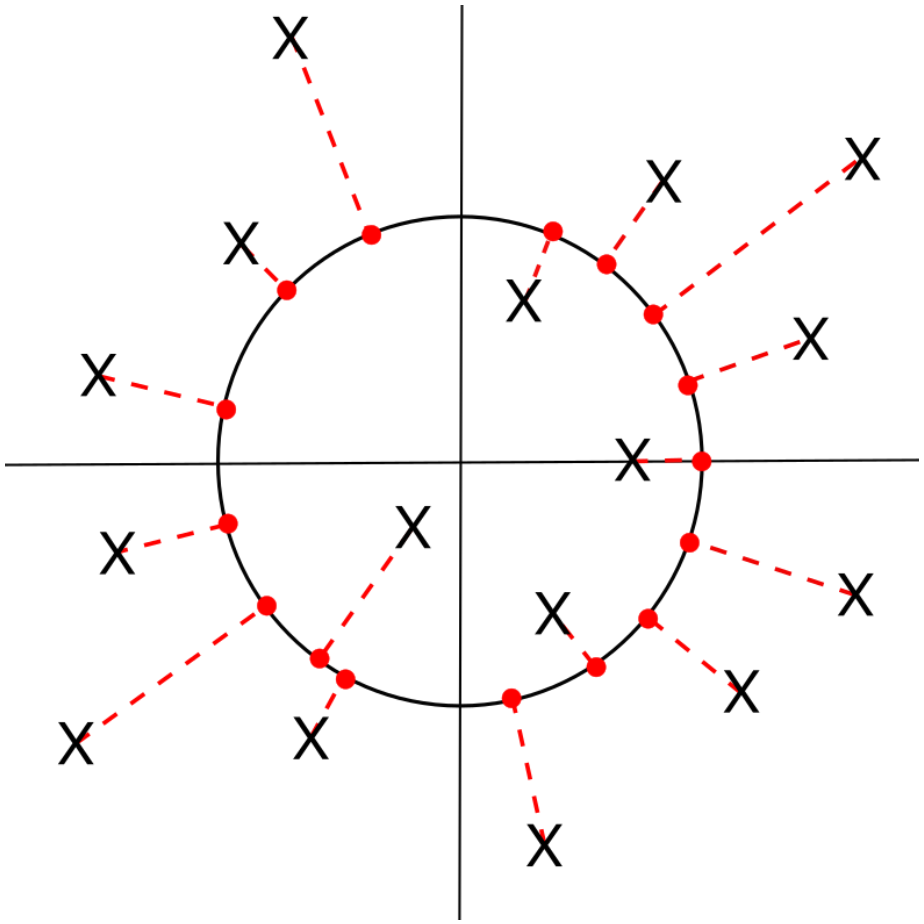


- *Masur*: angles are dense

Angles as a test of randomness



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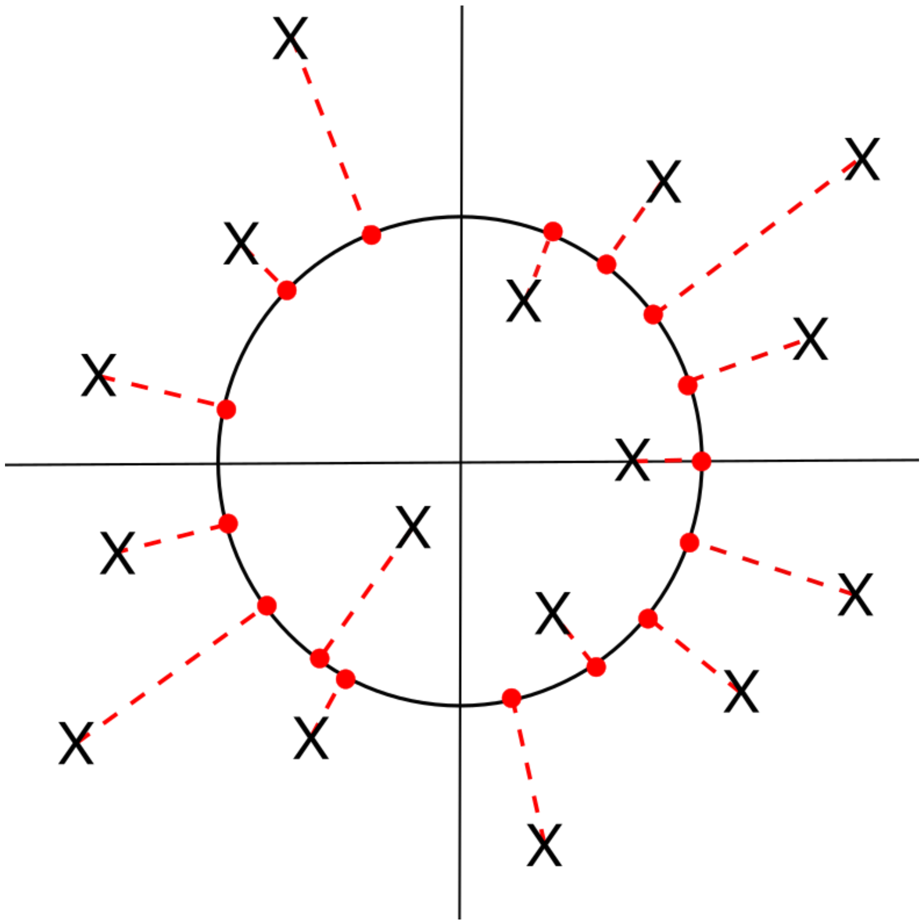
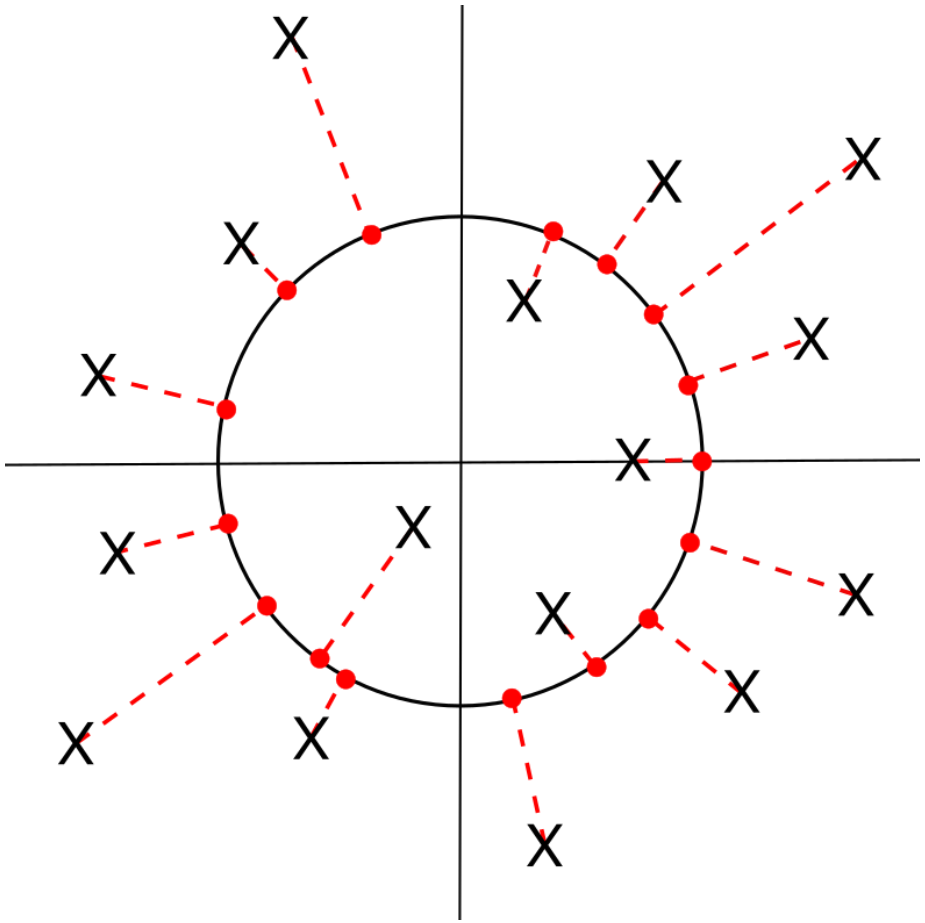


- *Masur*: angles are dense
- *Vorobets*: angles are ***equidistributed*** for almost every translation surface
- *Eskin-Marklof-Morris*: angles are ***equidistributed*** for covers of lattices surfaces



Upshot: Saddle connections appear to behave randomly at first glance.

A second test of randomness

A second test of randomness is to consider ***gaps*** of sequences.

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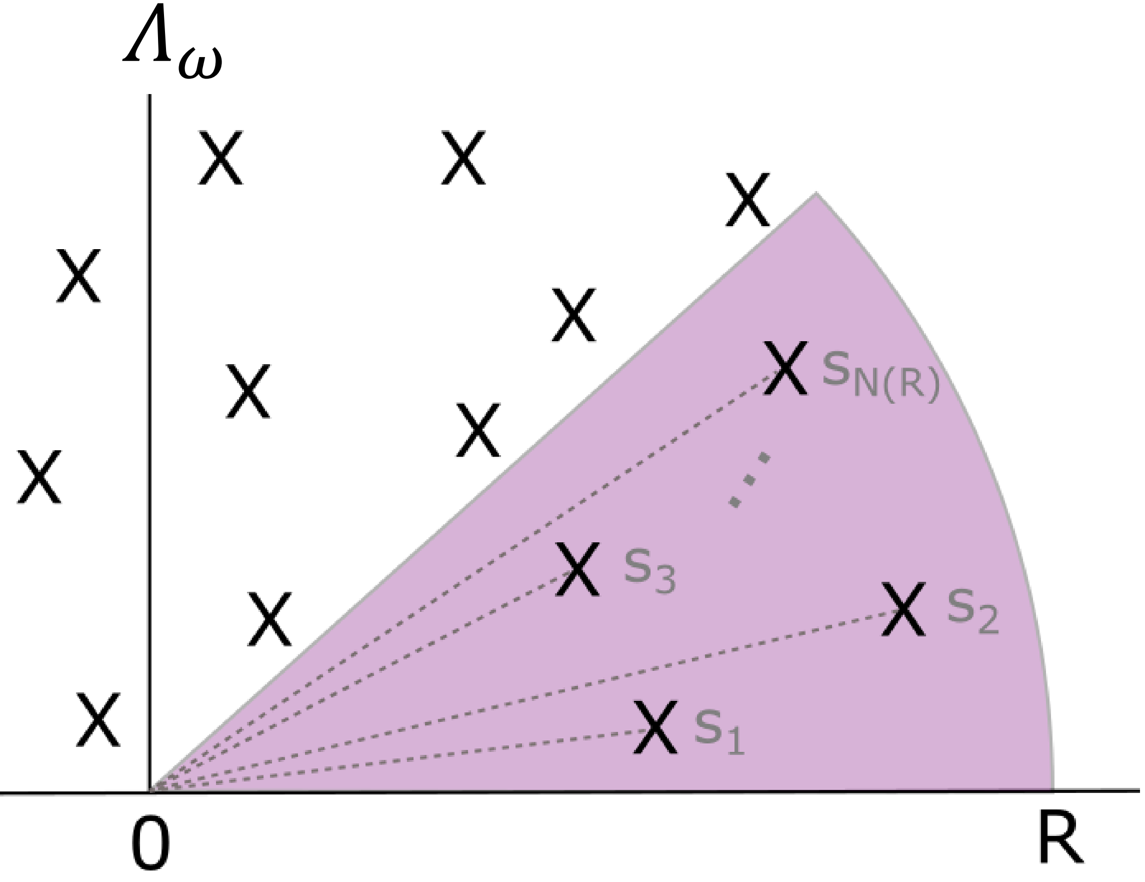
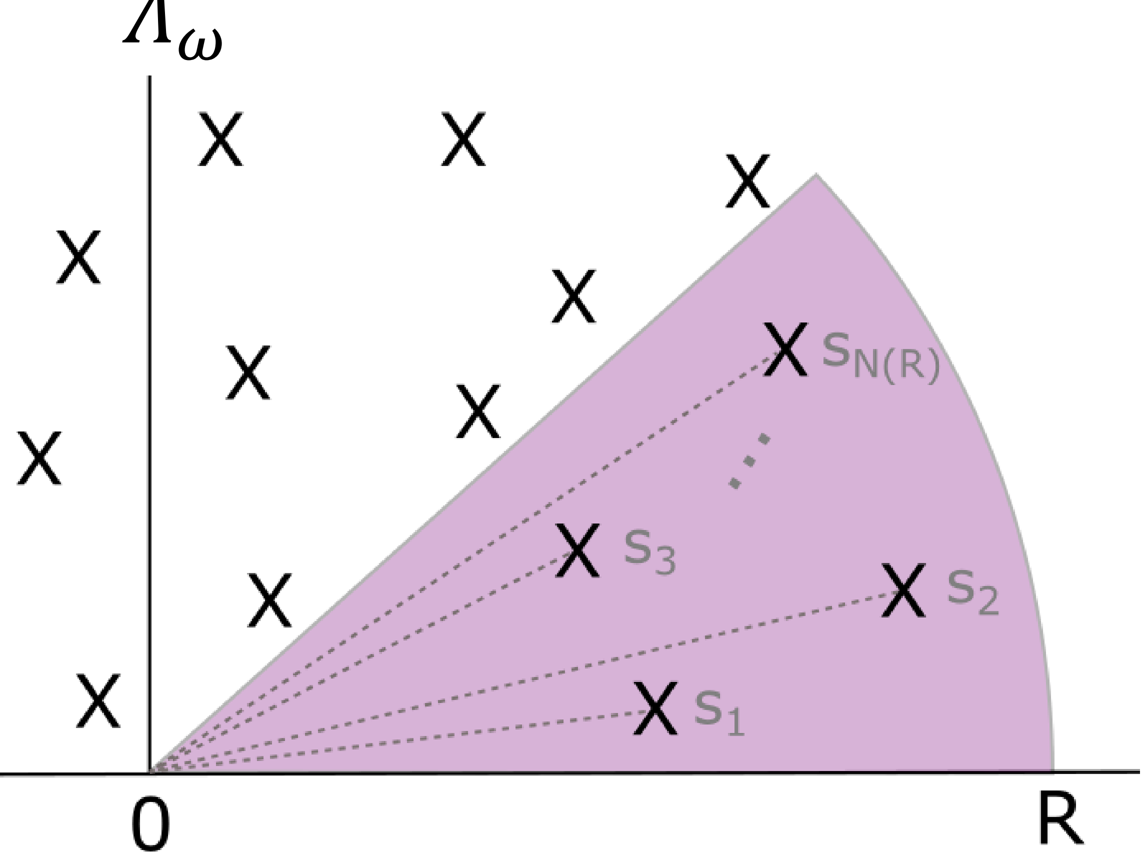
We consider slopes of saddle connections instead of angles.

Slopes of holonomy vectors



Let $Slopes^R(\Lambda_\omega)$ denote the slopes in an eighth sector up to length R.

Slopes of holonomy vectors

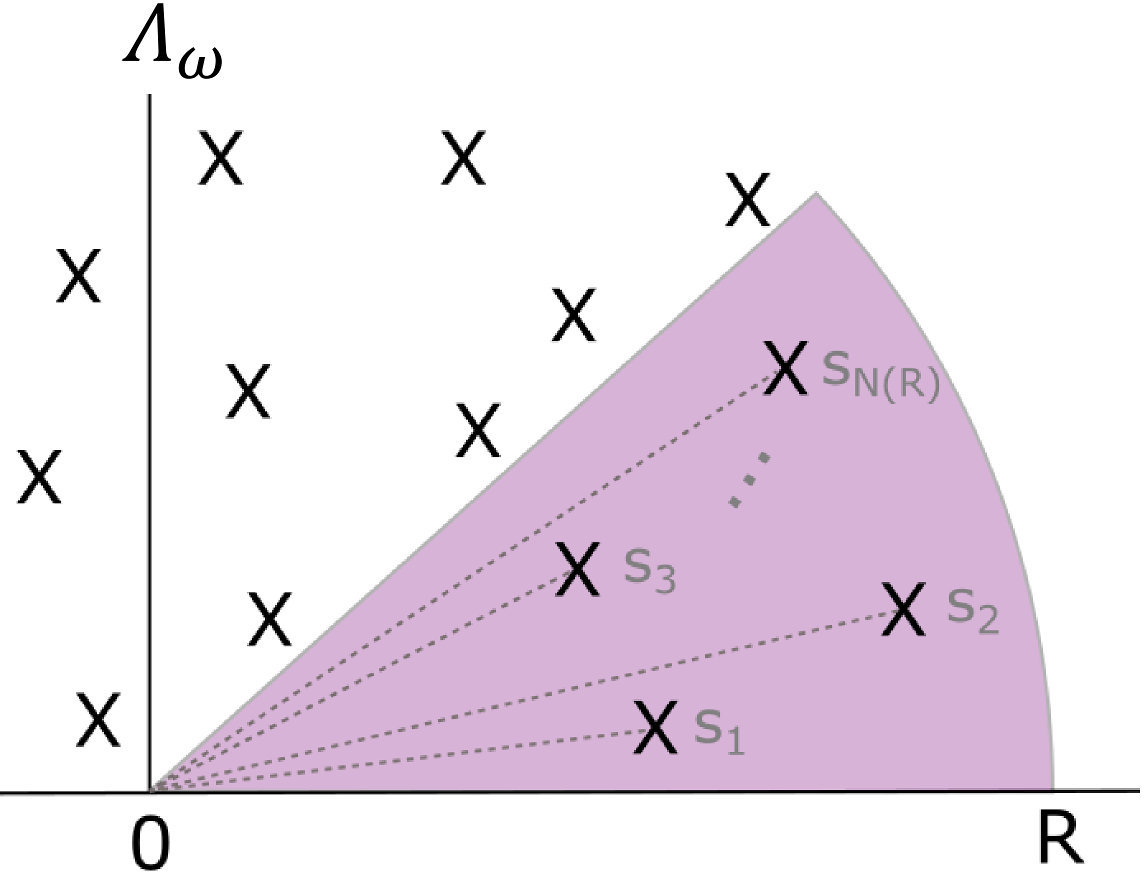


Let $Slopes^R(\Lambda_\omega)$ denote the slopes in an eighth sector up to length R.

$$Slopes^R(\Lambda_\omega) = \{s_0 = 0 < s_1 < \cdots < s_{N(R)}\}$$

where $N(R) = |Slopes^R(\Lambda_\omega)|$.

Slopes of holonomy vectors



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where $N(R) = |Slopes^R(\Lambda_\omega)|$.

Eskin-Masur showed $N(R) \sim \ R^2$.

Gaps of holonomy vectors

Consider the *gaps* of slopes

$$Gaps^R(\Lambda_\omega) = \{ \quad (s_i - s_{i-1}) | \; i = 1, \dots, N(R) \}$$

Gaps of holonomy vectors

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$$Gaps^R(\Lambda_\omega) = \{R^2(s_i - s_{i-1}) \mid i = 1, \ldots, N(R)\}$$

Gaps of holonomy vectors

Consider the *gaps* of slopes

$$Gaps^R(\Lambda_\omega) = \{R^2(s_i - s_{i-1}) |\ i = 1, \dots, N(R)\ \}$$

What can we say about the distribution of gaps?

Gap distribution

The *gap distribution* is given by

$$Gaps^R(\Lambda_\omega)$$

Gap distribution

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$$Gaps^R(\Lambda_\omega) \cap I$$

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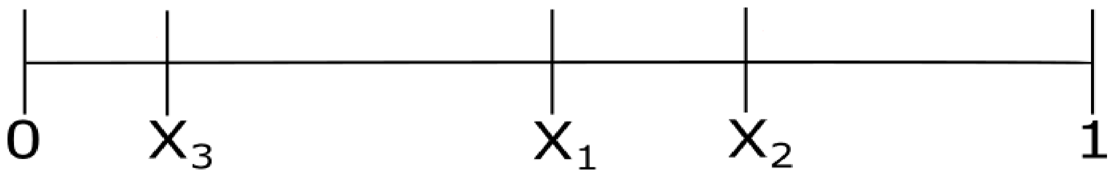
What can we say about this limit? What do we expect?

Context from probability

Suppose that $(X_i)_{i=1}^{\infty}$ are a sequence of IID random variables uniformly distributed on [0,1].



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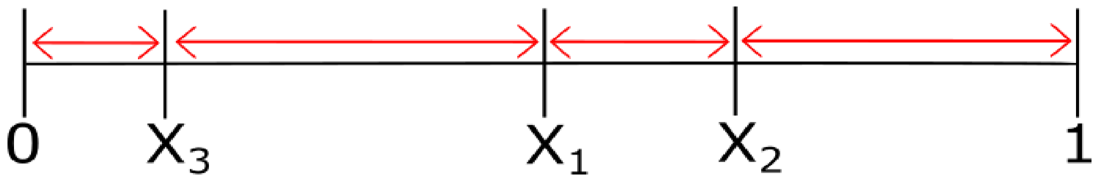

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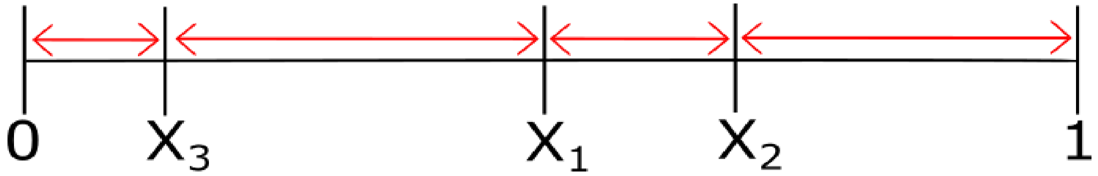

$$Gaps\{(X_i)_{i=1}^{n}\}$$

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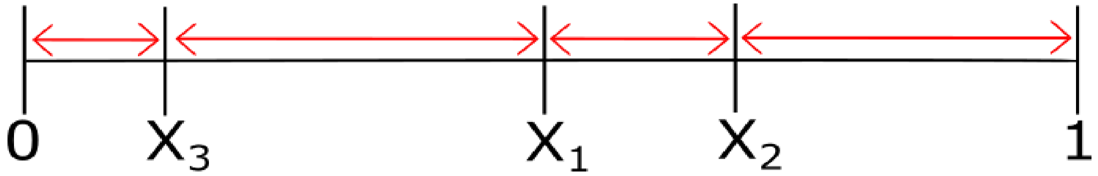
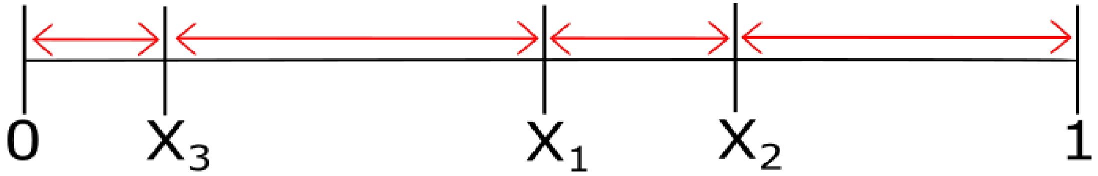


$$\frac{|Gaps\{(X_i)_{i=1}^{n}\} \cap I|}{n}$$

Context from probability

Suppose that $(X_i)_{i=1}^{\infty}$ are a sequence of IID random variables uniformly distributed on [0,1].

The associated gaps are **exponential**.

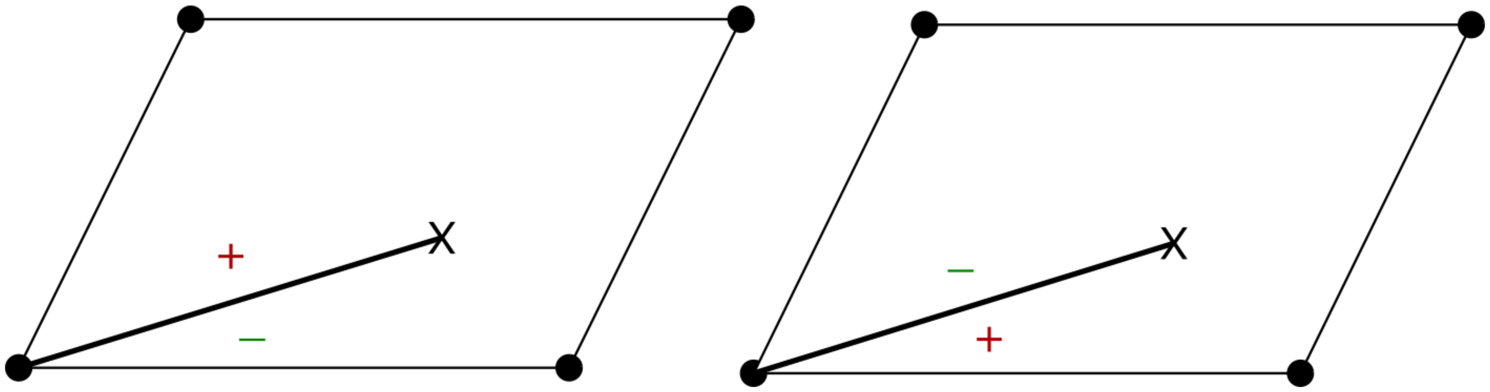


$$\frac{\left|Gaps\{(X_i)_{i=1}^{n}\}\cap I\right|}{n} \to \int_{\mathrm{I}} e^{-x}\,dx$$

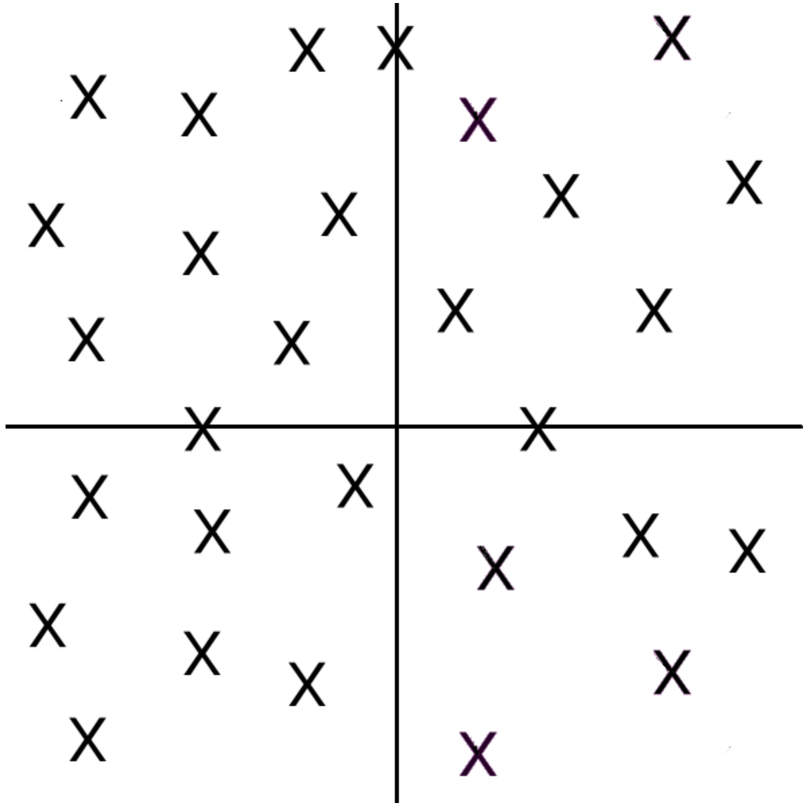
Theorem (S. 2020)

The gap distribution of almost every doubled slit torus is **not** exponential.

(X, ω)



Λ_ω

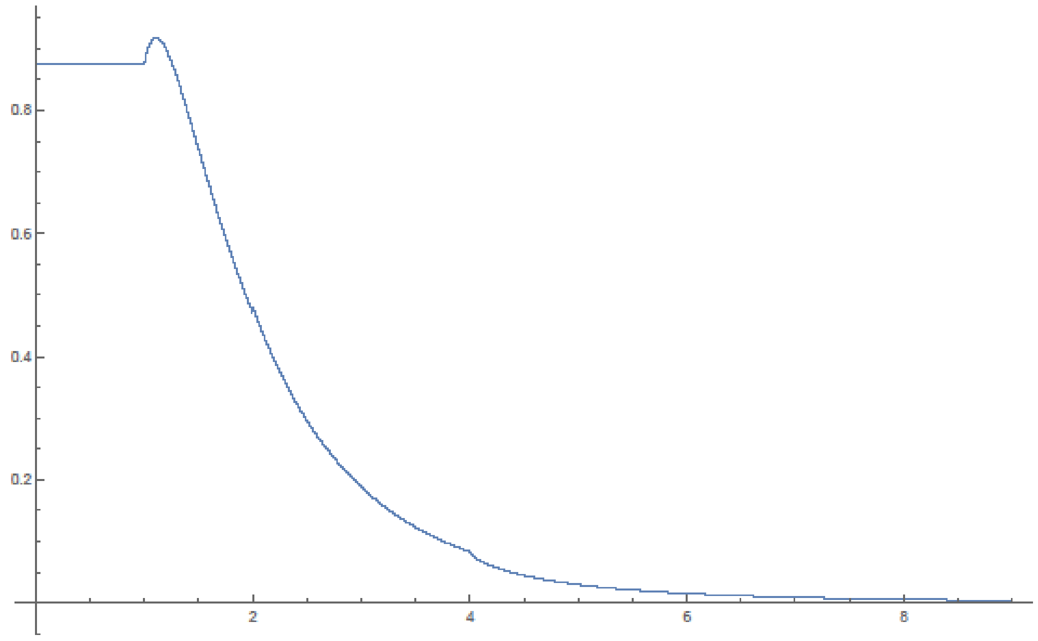


Theorem (S. 2020)

There exists a density function f so that

$$\lim_{R\to\infty} \frac{\left|Gaps^R(\Lambda_\omega)\cap I\right|}{N(R)} = \int_I f(x)\, dx$$

for almost every doubled slit torus.



Large gaps

The gap distribution has a *quadratic tail:*

$$\int_t^\infty f(x)\,dx \sim t^{-2}.$$

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Compare with the IID case:

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Thus, large gaps are unlikely, but still much more likely than the random case!

Small gaps

The gap distribution has *support at zero:*

$$\int_0^{\varepsilon} f(x)\, dx > 0$$

for every $\varepsilon > 0$.

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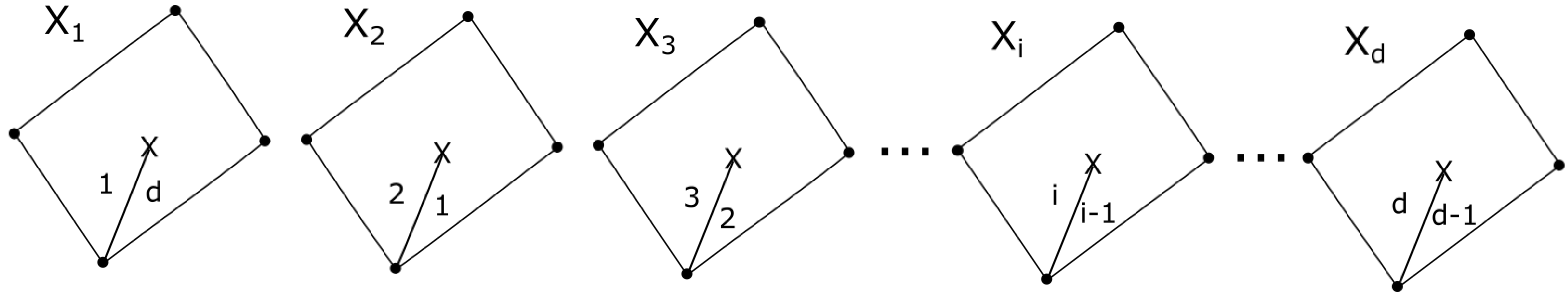
for every $\varepsilon > 0$.

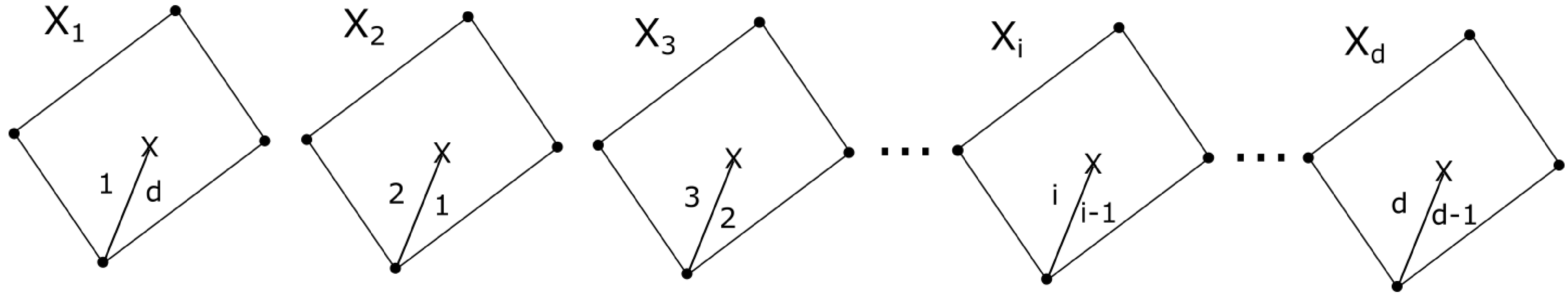
This is expected since doubled slit tori are not lattice surfaces.

Higher genus



These surfaces are called *symmetric torus covers*.

Higher genus



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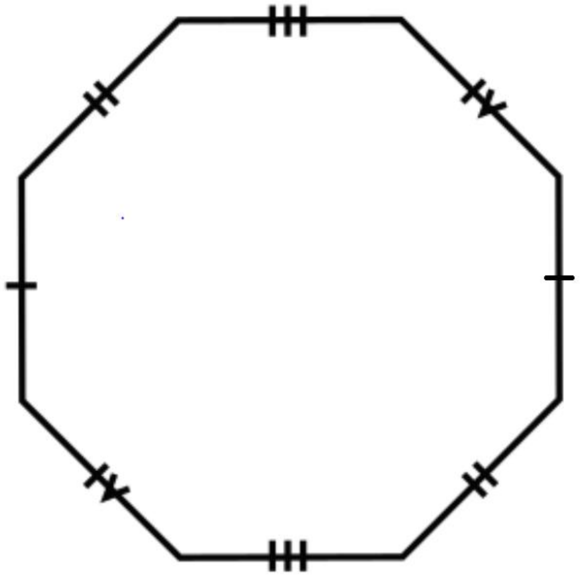
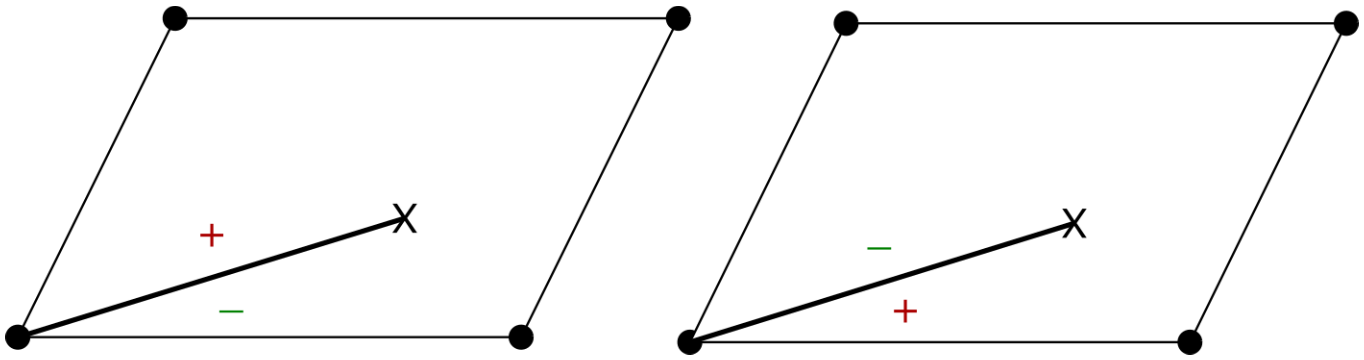
Symmetric torus covers have the same gap distribution as doubled slit tori.

Other results on gaps of translation surfaces

- Lattice surfaces (highly symmetric translation surfaces)

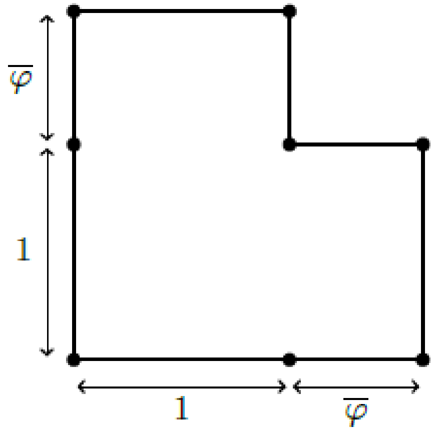


- Non-lattice surfaces



Gaps of lattice surfaces

- Athreya-Cheung (2014) - Torus
- Athreya-Chaika-Lelievre (2015) - Golden L
- Uyanik-Work (2016) - Regular octagon
- Taha (2020)- Gluing two regular (2n+1)-gons



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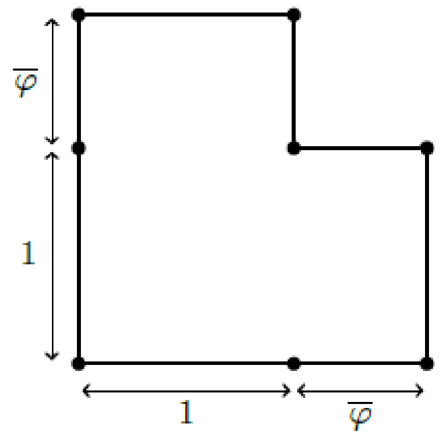
Characteristics of the gap distributions:

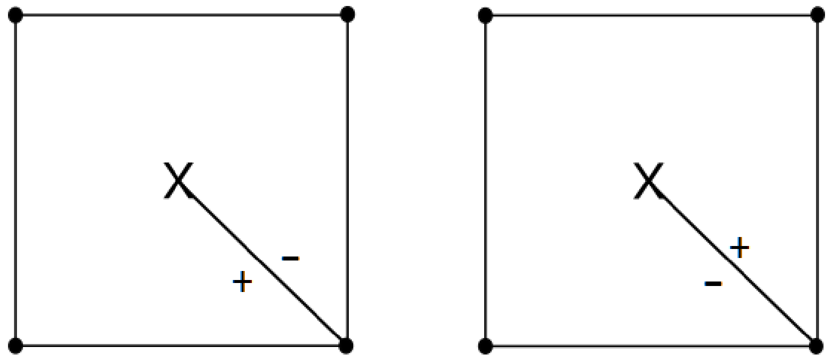
- No small gaps
- 2-dimensional parameter space
- Explicit gap distributions



Gaps of non-lattice surfaces

Athreya-Chaika (2012) – Generic translation surfaces

- Gap distribution exists for a.e. translation surface and is the same
- Non-explicit
- Small gaps characterize non-lattice surfaces



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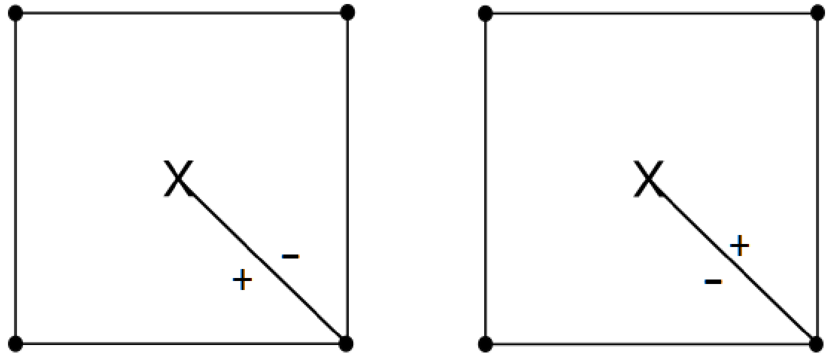

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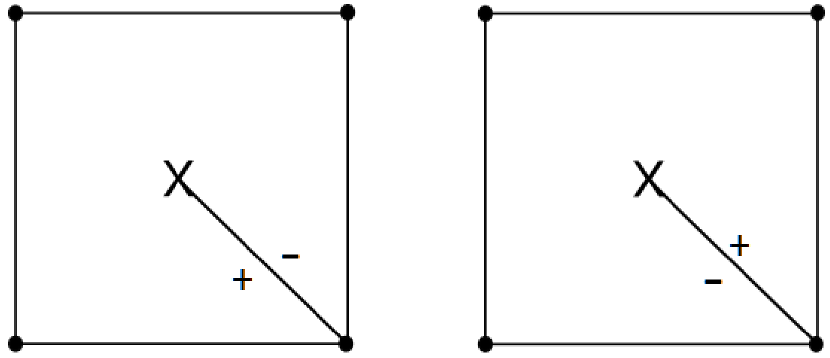
S. (2020) – Doubled slit tori

- Parameter space 4-dimensional
- First explicit gap distribution for non-lattice surface



Guiding philosophy

Questions about a *fixed* translation surface can be understood by considering the dynamics on the space of *all* translation surfaces.

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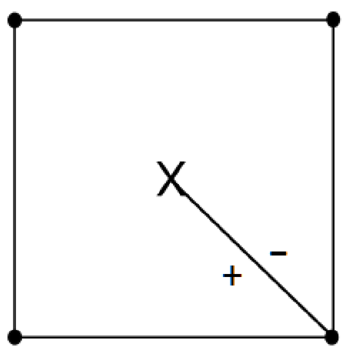
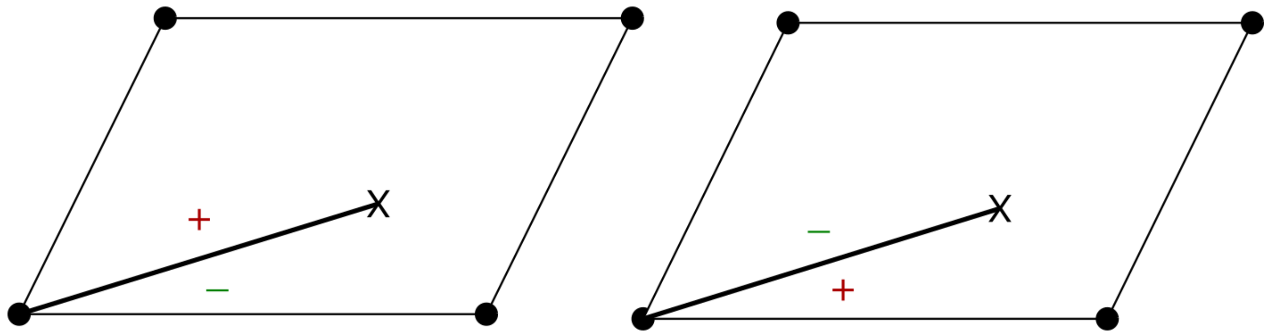
Gap distribution of a doubled slit torus → Dynamical question on the space of doubled slit tori

Translation surfaces $\mathcal{E}$

Let $\mathcal{E}$ denote the set of all doubled slit tori





The $SL(2, \mathbb{R})$-action

There is a "linear" action of $SL(2, \mathbb{R})$ on $\mathcal{E}$

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act on the polygon presentation

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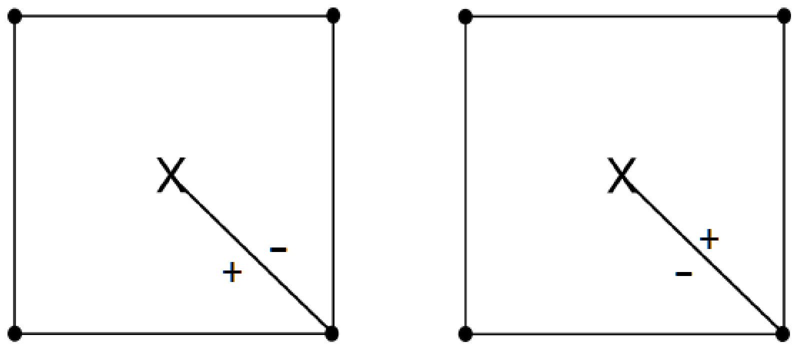

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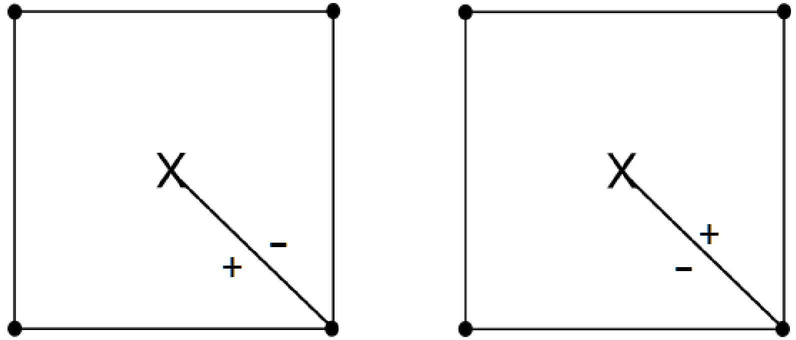

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

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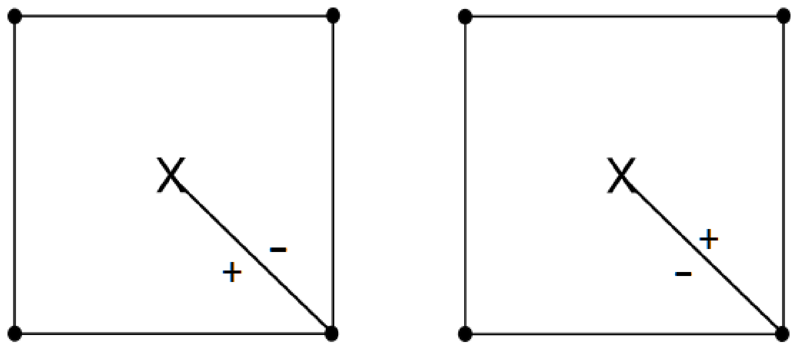
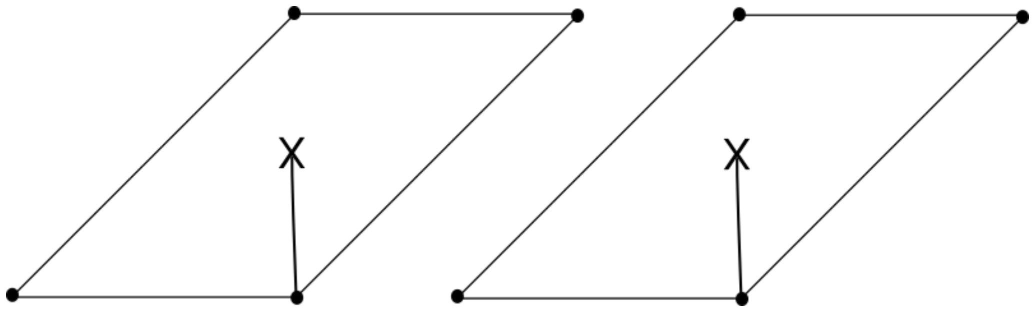


$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \longrightarrow$$



Horocycle flow

Consider the 1-parameter family

$$\left\{ h_u = \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix} : u \in \mathbb{R} \right\}$$

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- This subgroup is of interest because of how it changes slopes.

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$$\Downarrow$$

$$slope\left(h_u\begin{pmatrix}x\\y\end{pmatrix}\right) = slope\left(\begin{pmatrix}x\\y\end{pmatrix}\right) - u$$

Slopes

$$h_u\binom{x}{y} = \binom{x}{y - ux}$$

$$\downarrow$$

$$slope\left(h_u\binom{x}{y}\right) = slope\left(\binom{x}{y}\right) - u$$

In particular, ***slope differences*** are preserved!

Transversal for doubled slit tori

Consider the ***transversal*** for doubled slit tori

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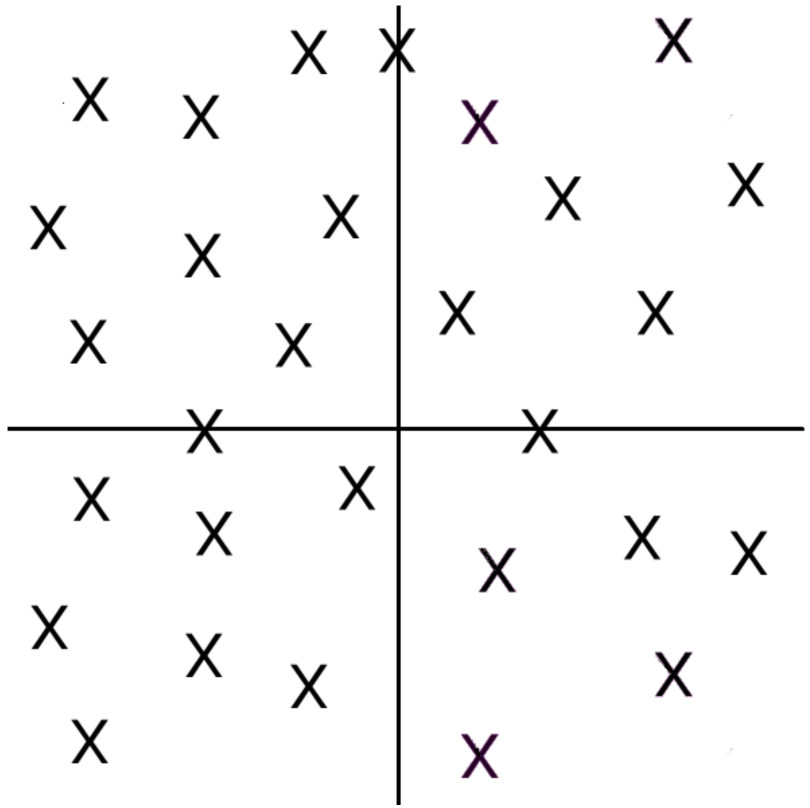
Λ_ω



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Key: slope gaps = return times to $\mathcal{W}$

- **First return time**:

  If $\omega \epsilon \mathcal{W}$, when is $h_u \omega \epsilon \mathcal{W}$?

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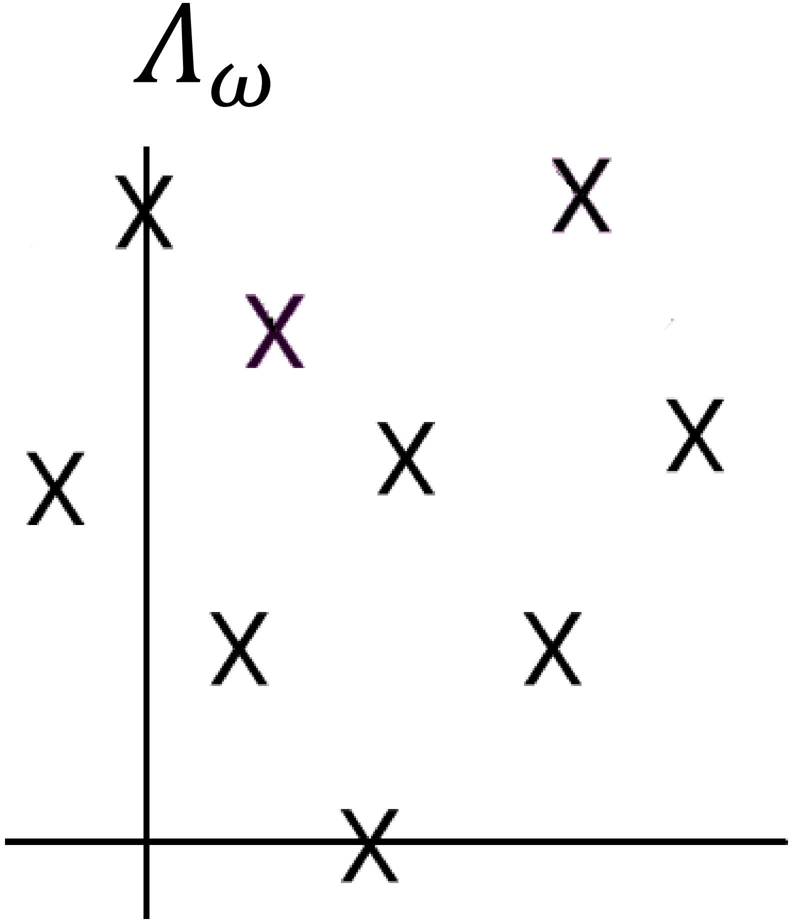

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Need a vector in Λ_ω with

$$h_u \binom{x}{y} = \binom{x}{y - ux}$$

short and horizontal.

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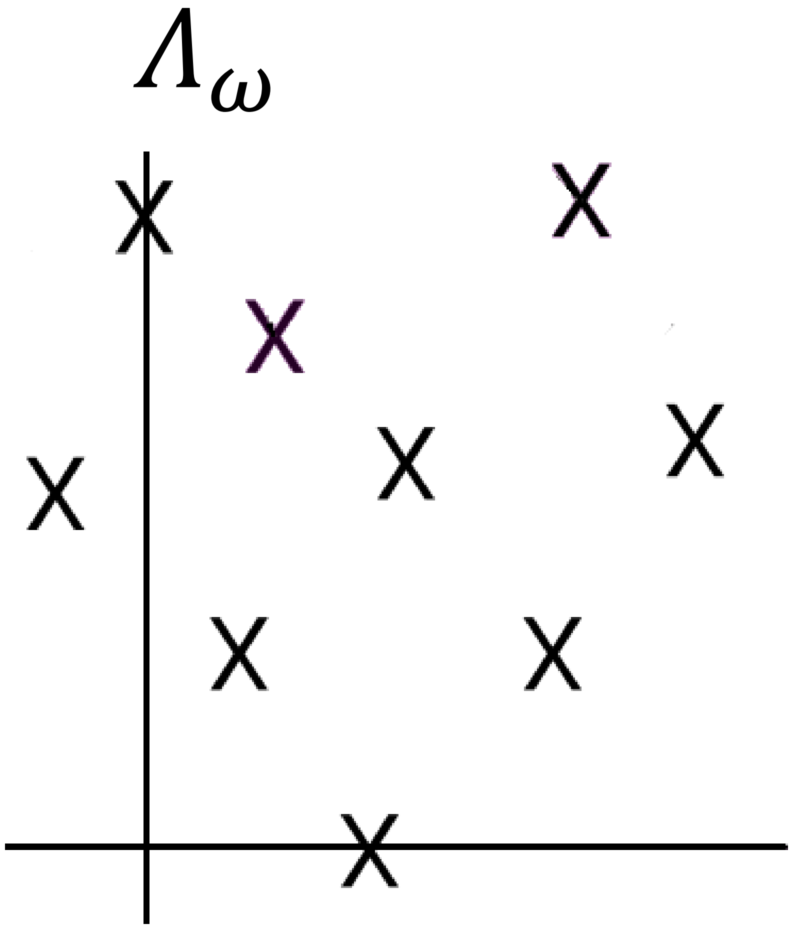

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  Need a vector in Λ_ω with

$$h_u \binom{x}{y} = \binom{x}{y - ux}$$

  short and horizontal.

- This happens is when

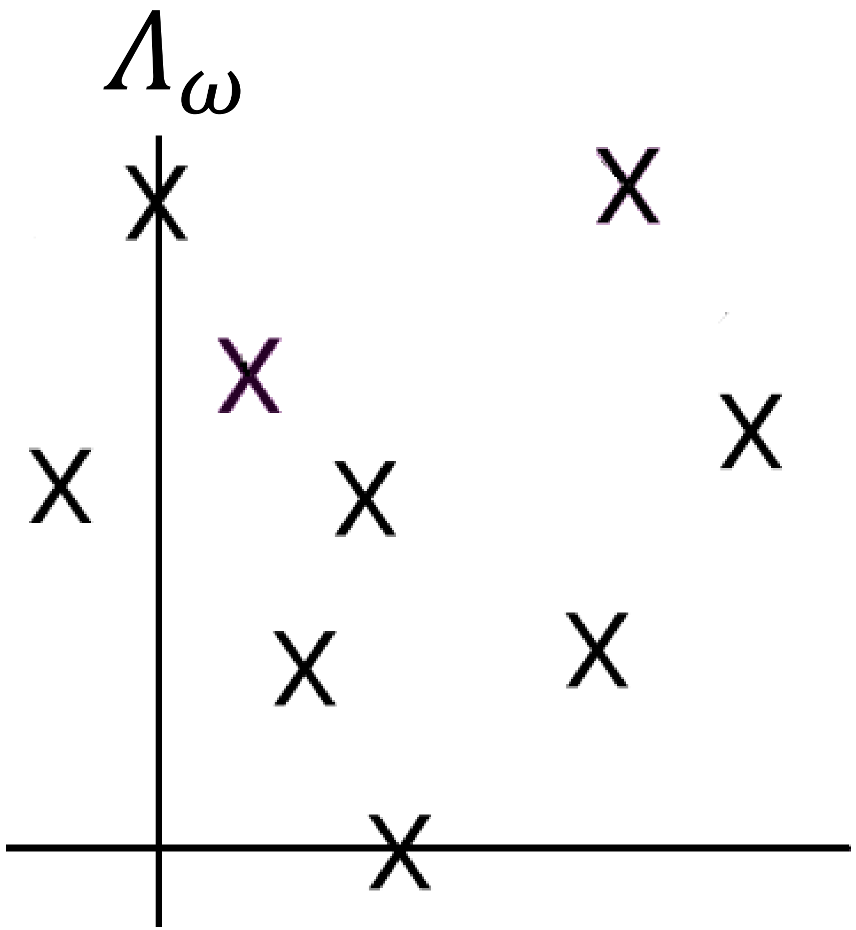
$$y - ux = 0 \Leftrightarrow u = \frac{y}{x}$$

So the ***first return time*** is a slope

So the ***first return time*** is a slope

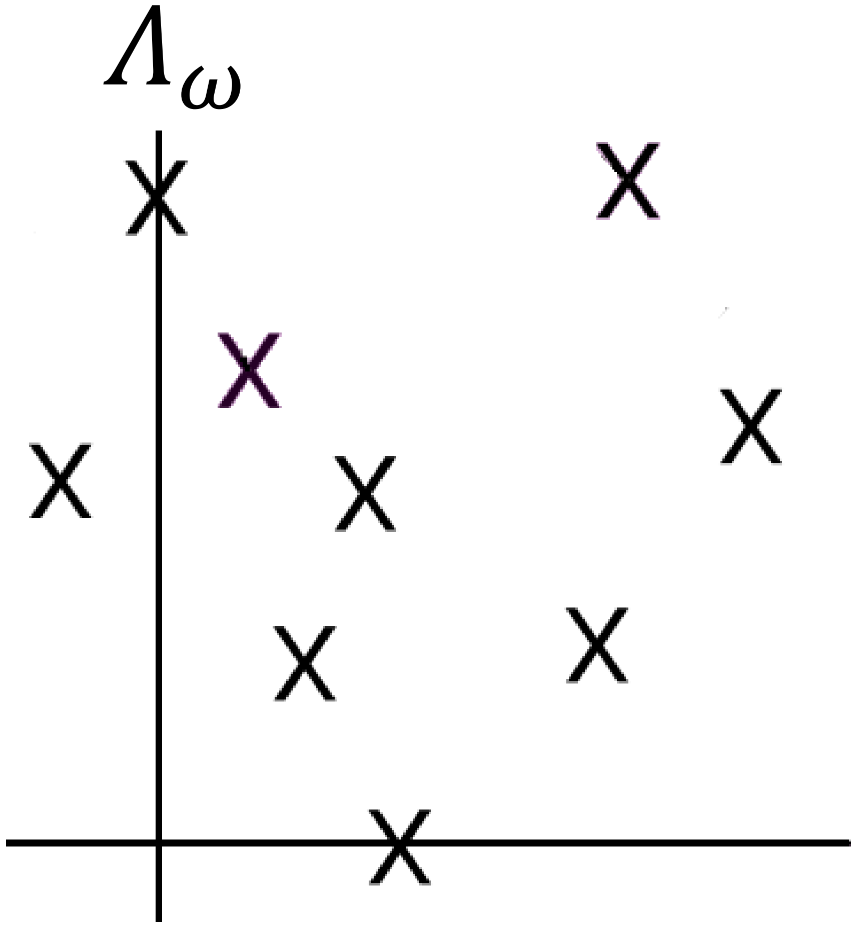
What about the second return time?

Second return time



Second return time = total time minus the first return time

Second return time



Second return time = total time minus the first return time

Hence, second return time is a slope difference.

Formalizing the key idea

Let R denote the *return time*

Let T denote the *return map*

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↓

$$s_{i+1} - s_i = R\left(T^i(\omega)\right)$$

Slope gaps as a dynamical question

$$\frac{|Gaps^N(\Lambda_\omega) \cap I|}{N}$$

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$$\rightarrow \mu\{\omega \epsilon \mathcal{W} \,|R(\omega) \in I\}$$

So next steps:

- parametrize $\mathcal{W}$
- find return map in coordinates

Thank you!

Special thanks to:

- Dr. Jayadev Athreya (My advisor)
- UC San Diego Group Actions Seminar