Topological dynamics beyond Polish groups

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The universal minimal flow of a topological group

Given a topological group G, a *G-flow* is a compact space X together with a continuous action of G on X. A G-flow is *minimal* if each of its orbits is dense, or equivalently, if it contains no proper subflows. If X, Y are two G-flows a continuous map $f : X \to Y$ is a *G-map* if it commutes with the action. If Y is minimal then f has to be onto.

Fact (Ellis '60)

For each topological group G there exists a universal minimal flow $M(G)$ which is unique up to isomorphism.

First facts on the UMF

If G is *compact*, then $M(G)$ is G itself with the natural action by translation. Indeed, G is minimal and if X is a minimal G-flow, the map $\rho_x(g) = g \cdot x$, with $x \in X$, is a G-map.

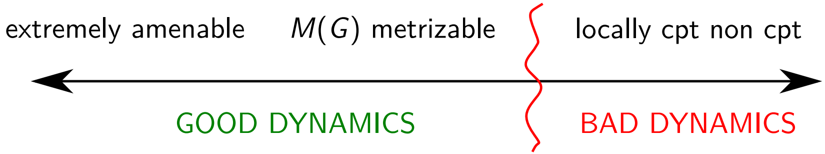
If G is *infinite discrete* then it acts freely on $M(G)$, which is a non-metrizable subset of the space βG of ultrafilters on G.

By a Theorem of Veech, also for G *locally compact* the action on $M(G)$ is free. If G is not compact then $M(G)$ is again a non-metrizable space.

Extreme amenability

There are large groups G such that $|M(G)| = 1$. For example the unitary group $U(\ell^2)$ with the strong operator topology (Gromov-Milman). Such groups are called *extremely amenable*.

Recall: A topological group is *amenable* if every G-flow admits an invariant probability measure.



Polish groups and non-archimedean groups

A topological group is *Polish* if it is separable and completely metrizable.

Examples: second countable locally compact groups, Homeo(X) of a compact metrizable space X, the group Sym($\mathbb{N}$) of all permutations of a countable set.

Definition: A topological group is *non-archimedean* if the identity has a basis consisting of open subgroups.

Fact: The Polish non-archimedean groups are exactly the closed subgroups of Sym($\mathbb{N}$). They are also exactly the automorphism groups of countable (ω-homogeneous) structures.

Examples: Aut($\mathbb{Q}, <$), Aut(R) of the Rado graph.

Fact: Aut($\mathbb{Q}, <$) is extremely amenable (Pestov '98). This is equivalent to the classic Ramsey theorem.

The case of Aut($\boldsymbol{K}$)

For a relational structure $\boldsymbol{K}$, let Age($\boldsymbol{K}$) be the class of finite substructures of $\boldsymbol{K}$. Then $\boldsymbol{K}$ is ω-*homogeneous* if any isomorphism of finite substructures of $\boldsymbol{K}$ extends to an automorphism of $\boldsymbol{K}$. A countable ω-homogeneous structure is a *Fraïssé* structure.

Fact (Kechris-Pestov-Todorcevic)

Let $\boldsymbol{K}$ be a Fraïssé structure. Then Aut($\boldsymbol{K}$) *is extremely amenable if and only if* Age($\boldsymbol{K}$) *has the Ramsey property.*

Fact (Zucker)

*Let $\boldsymbol{K}$ be a Fraïssé structure. Then M(*Aut($\boldsymbol{K}$)*) is metrizable if and only if each $A \in$* Age($\boldsymbol{K}$) *has finite Ramsey degrees if and only if* Age($\boldsymbol{K}$) *admits an appropriate expansion class with the Ramsey property. In such a case, M(*Aut($\boldsymbol{K}$)*) has a concrete representation as a space of expansions of $\boldsymbol{K}$.*

Metrizability of the UMF of Polish groups

Fact (Ben Yaacov-Melleray-Tsankov; Bartošová-Zucker; Jahel-Zucker)

Let G be a Polish group. TFAE:

1. *$M(G)$ is metrizable.*
2. *The UEB metric on $M(G)$ is compatible.*
3. *$\beta\mathbb{N}$ does not embed in $M(G)$.*
4. *There is a closed extremely amenable subgroup $H \leq G$ such that the completion of G/H is a minimal G-flow (equiv. is the UMF).*
5. *For any G-flow X, the set $AP(X)$ is closed, thus a subflow.*

Definition: If X is a G-flow, the set $AP(X) \subseteq X$ of *almost periodic points* is the union of the minimal subflows of X.

Does there exist a meaningful extension of this dividing line beyond Polish?

The first step outside Polish

Fact (Zucker)

Let $\boldsymbol{K}$ be a Fraïssé structure. Then $M(\mathrm{Aut}(\boldsymbol{K}))$ is metrizable if and only if $\mathrm{Age}(\boldsymbol{K})$ admits an appropriate expansion class with the Ramsey property. In such a case, $M(\mathrm{Aut}(\boldsymbol{K}))$ has a concrete representation as a space of expansions of $\boldsymbol{K}$.

Fact (Bartošová)

Let $\boldsymbol{K}$ be a ω-homogeneous structure. If $\mathrm{Age}(\boldsymbol{K})$ admits an appropriate expansion class with the Ramsey property then $M(\mathrm{Aut}(\boldsymbol{K}))$ has a concrete representation as a space of expansions of $\boldsymbol{K}$.

Example: $M(Sym(\kappa)) = LO(\kappa)$ is the space of linear orders on κ.

Theorem (B.-Zucker) Under the above conditions, $AP(X)$ is closed for any $\mathrm{Aut}(\boldsymbol{K})$-flow X.

CAP groups

Definition (B.-Zucker)

A topological group G is CAP if $AP(X)$ is closed for every G-flow X.

Recall:

- $AP(X)$ is the union of the minimal subflows of X.
- A subflow $Y \subseteq X$ is minimal if $\overline{Gy} = Y$ for all $y \in Y$.

Definition: Let $x \sim_{AP(X)} y \iff \overline{Gx} = \overline{Gy}$ be the equivalence relation on $AP(X)$ whose equivalence classes are the minimal flows of which $AP(X)$ is composed.

Next goal: define a canonical uniformity on $M(G)$.

Uniform spaces

A *uniform structure* $\mathcal{U}$ on a set X is a filter of supersets of the diagonal $\Delta \subseteq X \times X$, called *entourages*, such that:

- for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ with $V^2 = \{(x, z) \mid \exists y\, (x, y), (y, z) \in V\} \subseteq U$,
- if $U \in \mathcal{U}$, then $U^{-1} = \{(y, x) \mid (x, y) \in U\} \in \mathcal{U}$,
- $\bigcap_{U \in \mathcal{U}} U = \Delta$.

Topological groups admit a canonical compatible uniform structure, the *right uniformity*, which is generated by

$$\left\{(g, h) \in G \times G \;\middle|\; gh^{-1} \in U\right\},$$

for U an open neighborhood of the identity.

Compact spaces admit a *unique* compatible uniform structure: all neighborhoods of the diagonal.

The Samuel compactification

A function $f : X \to Y$ is *uniformly continuous* if for each entourage V of Y there is an entourage U of X such that $(f(x), f(y)) \in V$ for all $(x, y) \in U$.

The *Samuel compactification* $S(G)$ is a G-flow which densely embeds G and has the following universal property: if X is a uniform space, each uniformly continuous $f : G \to X$ uniquely extends to a continuous $\widehat{f} : S(G) \to X$.

$$\begin{array}{ccc} S(G) & & \\ \uparrow & \overset{\widehat{f}}{\searrow} & \\ G & \xrightarrow[f]{} & X \end{array}$$

Suppose X is a minimal G flow and $f = \rho_x : g \mapsto g \cdot x$ for some $x \in X$. Then $\widehat{\rho_x}_{|M}$ is a G-map for any minimal subflow $M \subseteq S(G)$.

Fact: Each minimal subflow of $S(G)$ is isomorphic to $M(G)$.

The UEB uniformity

A set H of functions $G \to [0,1]$ is *uniformly equicontinuous* if for every $\epsilon > 0$ there is $U \ni_{op} 1_G$ so that for any $g, h \in G$ with $gh^{-1} \in U$, we have $|f(g) - f(h)| < \epsilon$ for each $f \in H$.

Definition: The *UEB uniformity* on $S(G)$ is given by the basic entourages, for $H \subseteq \mathcal{C}(G, [0,1])$ uniformly equicontinuous and $\epsilon > 0$:

$$[H, \epsilon] = \Big\{(p,q) \in S(G) \times S(G) : |\widehat{f}(p) - \widehat{f}(q)| < \epsilon \text{ for all } f \in H\Big\}.$$

The restriction of this uniformity to $M(G) \subseteq S(G)$ does not depend on the choice of minimal subflow.

When G is Polish, the UEB uniformity is actually a metric which is lower semi-continuous on $\mathrm{M}(G)$. We can define it directly as:

$$d(p,q) = \sup\Big\{|\widehat{f}(p) - \widehat{f}(q)| : f \in Lip(G)\Big\}$$

UEB topo-uniformity

In general the UEB uniformity on $M(G)$ is not compatible with the compact topology.

Theorem (B.-Zucker)

The space $(M(G), \tau)$ together with the UEB uniformity form a topo-uniform space, that is:

- *each $(\tau \times \tau)$-open neighborhood of the diagonal is an entourage,*
- *the uniformity has a basis of $(\tau \times \tau)$-closed entourages.*

Characterization theorem

Theorem (B.-Zucker)

Let G be a topological group. TFAE:

1. *G is CAP.*
2. *G is CAP and $x \sim_{AP(X)} y \iff \overline{Gx} = \overline{Gy}$ is closed for each G-flow X.*
3. *The UEB uniformity on $\mathrm{M}(G)$ is compatible with the compact topology.*
4. $\mathrm{M}(G \times G) \cong \mathrm{M}(G) \times \mathrm{M}(G)$.

Question: Are the above equivalent to "$AP(S(G))$ is closed"?

It would follow from a positive answer to the ambitability/unique amenability question (Pachl):

If G admits a *unique* G-invariant probability measure on any flow with a dense orbit, is G precompact?

Which groups are CAP?

Theorem (B.-Zucker)

1. *Every precompact group is CAP.*
2. *Every group with metrizable UMF is CAP.*
3. *The class of CAP groups is closed under quotients, group extensions, inverse limits and products.*
4. *If $\boldsymbol{K}$ is a ω-homogeneous structure, then $\mathrm{Aut}(\boldsymbol{K})$ is CAP if and only if $\mathrm{Age}(\boldsymbol{K})$ has finite Ramsey degrees.*
5. *Locally compact not compact groups are not CAP.*

Theorem (B.-Zucker)

If G_i is CAP for all $i \in \mathcal{I}$, then

$$M\left(\prod_{i\in\mathcal{I}} G_i\right) = \prod_{i\in I} M(G_i).$$

Scattered spaces

A topological space is *scattered* if it does not contain any nonempty perfect subspace.

Fact (Gheysens '20+)

If X is scattered the topology of pointwise convergence agrees with the topology of discrete pointwise convergence on $\mathrm{Homeo}(X)$*.*

Therefore $\mathrm{Homeo}(X)$ embeds in $\mathrm{Sym}(|X|)$.

Any ordinal with the order topology is scattered, in particular ω_1.

Homeo(ω_1) and its UMF

Fact (Gheysens '20+)

Homeo(ω_1) *is amenable, Roelcke-precompact, not Baire, and admits no nontrivial homomorphism to any metrizable group.*

Fact (Gheysens '20+)

The closure of Homeo(ω_1) *in* Sym(ω_1) *is isomorphic to* Sym$(\omega_1)^{\omega_1}$.

Theorem (B.-Zucker)

Homeo(ω_1) *is CAP and* $M(\text{Homeo}(\omega_1)) = LO(\omega_1)^{\omega_1}$.

Missing converses

Theorem (B.-Zucker)

If G is not CAP then $\beta\mathbb{N}$ embeds in $M(G)$. If G is CAP, then there is a $\supseteq$-monotone and cofinal map from $\mathcal{N}_G$ to $\mathrm{Nbhd}(\Delta_{M(G)})$.

Question: Is there a condition on the "size" of $M(G)$ which is equivalent to being CAP?

Theorem (B.-Zucker)

If G admits a closed extremely amenable subgroup H such that the completion of G/H is a minimal G-flow, then G is CAP and $M(G)$ is the completion of G/H.

Question: Does the converse hold for complete groups G?

For instance: if $\boldsymbol{K}$ is an uncountable, ω-homogeneous graph which embeds every finite graph, does there exist a linear order on $\boldsymbol{K}$ so that $(\boldsymbol{K}, <)$ is also ω-homogeneous? Here: $G = \mathrm{Aut}(\boldsymbol{K})$, $H = \mathrm{Aut}(\boldsymbol{K}, <)$.

Thank you!