An Introduction to the f-invariant

Nov. 17, 2020

Setting and notation:

- G countably infinite group
- (X, μ) standard probability space
- $G \curvearrowright (X, \mu)$ measure-preserving
- Write n for $\{0, 1, 2, \ldots, n-1\}$
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Definition / Example

The *Bernoulli n-shift over G* is $G \curvearrowright (n^G, u_n^G)$

Definition

$G \curvearrowright (X, \mu)$ *factors onto* $G \curvearrowright (Y, \nu)$ if there is a measurable map $\phi : X \to Y$ satisfying:

- $\phi_*(\mu) = \nu$
- $\phi(g \cdot x) = g \cdot \phi(x)$ for almost-every x and every g

If additionally ϕ is injective on a conull set then it is an *isomorphism*.

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- ϕ is a continuous everywhere 2-to-1 surjection
- ϕ commutes with the action of F_2
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Question (Ornstein–Weiss, 1987)

Are the Bernoulli shifts $F_2 \curvearrowright (2^{F_2}, u_2^{F_2})$ and $F_2 \curvearrowright (4^{F_2}, u_4^{F_2})$ isomorphic?

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Theorem (Kolmogorov–Sinai, 1958)

$h_{\mathbb{Z}}$ is an isomorphism invariant, it is non-increasing under factors, and $h_{\mathbb{Z}}(n^{\mathbb{Z}}, u_n^{\mathbb{Z}}) = \log(n)$

Fix an action $G \curvearrowright (X, \mu) = G \curvearrowright (X, \mathcal{B}, \mu)$

Definition

A partition $\mathcal{P}$ is *generating* if $\sigma\text{-alg}(\{g \cdot P : g \in G,\ P \in \mathcal{P}\}) = \mathcal{B}$

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Basic Fact

For every countable set A, there is a 1-to-1 correspondence between

- A-labeled generating partitions $\mathcal{P} = \{P_a : a \in A\}$
- isomorphisms ϕ mapping to $G \curvearrowright (A^G, \phi_*(\mu))$

Proof Sketch

($\rightarrow$) Set $\phi(x)(g) = a$ when $g^{-1} \cdot x \in P_a$
($\leftarrow$) Define $P_a = \{x \in X : \phi(x)(1_G) = a\}$

Fix a rank r free group $G = \langle S \rangle$, $|S| = r$, and fix $G \curvearrowright (X, \mu)$

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The *mutual information* of two finite partitions $\mathcal{P}$, $\mathcal{Q}$ is

$$I(\mathcal{P}, \mathcal{Q}) = \mathrm{H}(\mathcal{P}) + \mathrm{H}(\mathcal{Q}) - \mathrm{H}(\mathcal{P} \vee \mathcal{Q})$$

For a finite partition $\mathcal{P}$ set

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If $\mathcal{Q} \leq \mathcal{P}$ and $t \in S \cup S^{-1}$, call $\mathcal{P} \vee t \cdot \mathcal{Q}$ a *simple splitting* of $\mathcal{P}$

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Definition

$\mathcal{P}'$ is a *splitting* of $\mathcal{P}$ if there are $\mathcal{P}_i$ $(1 \leq i \leq n)$ with $\mathcal{P}_1 = \mathcal{P}$, $\mathcal{P}_n = \mathcal{P}'$, and $\mathcal{P}_{i+1}$ a simple splitting of $\mathcal{P}_i$

Definition (Bowen, 2010)

The *f-invariant* of a finite partition $\mathcal{P}$ is

$$f(\mathcal{P}) = \inf_{n \in \mathbb{N}} F(\bigvee_{g \in B_n} g \cdot \mathcal{P}),$$

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Theorem (Bowen 2010)

If $\mathcal{P}$ and $\mathcal{Q}$ are generating partitions then $f(\mathcal{P}) = f(\mathcal{Q})$. The common value (when defined) is called the *f-invariant* of the action and denoted $f(X, \mu)$.

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Proof Outline

- If $\mathcal{P}$ and $\mathcal{P}'$ share a common splitting then $f(\mathcal{P}) = f(\mathcal{P}')$
- Prove $\mathcal{Q}$ can be approximated by such $\mathcal{P}'$ above
- f is upper-semicontinuous, so $f(\mathcal{Q}) \geq f(\mathcal{P})$

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$G_s = \{\text{words } g \in G \text{ that don't start with s}\}$
μ is *Markov* if for every $s \in S \cup S^{-1}$

> *the distribution of $x(s)$ conditioned on $x \restriction G_s$ is equal to the distribution of $x(s)$ conditioned on $x(1_G)$*

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Theorem (Bowen 2010)

Let $\mathcal{P} = \{P_a : a \in A\}$ where $P_a = \{x \in A^G : x(1_G) = a\}$.
If μ is Markov then $f(A^G, \mu) = F(\mathcal{P}) = \mathrm{H}(\mathcal{P}) - \sum_{s \in S} I(\mathcal{P}, s \cdot \mathcal{P})$

The f-invariant has many nice properties...

- Does not depend on the choice of generating set S of G (Bowen 2010)
- There is a notion of relative f-invariant satisfying $f(\mathcal{P}) = f(\mathcal{Q}) + f(\mathcal{P}|\mathcal{Q})$ (Bowen 2010)
- When you restrict an action to a finite-index subgroup the f-invariant scales by the index (S 2014)
- The f-invariant satisfies an ergodic decomposition formula (S 2016)
- The f-invariant (Bowen 2010) and relative f-invariant (Shriver 2020) can be defined using sequences of finite random graphs
- Is related to sofic entropy, and when $G = \mathbb{Z}$, $f(X, \mu) = h_{\mathbb{Z}}(X, \mu)$
- In some cases satisfies the Juzvinskii addition formula (Bowen–Gutman 2014)

And a few strange features

- Can increase under factor maps (Ornstein–Weiss example)
- Can be negative
  If X finite and $G \curvearrowright X$ transitive then
  $f(X, \mu) = (1 - r) \log |X|$
- Can be $-\infty$.
  In fact, every action on a compact Riemannian manifold by diffeomorphisms has f-invariant $-\infty$ (Bowen–Gutman 2014)

Thank you!