Isometric Orbit Equivalence

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Talk at UCSD Group Actions Seminar

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Framework

- Probability measure preserving (pmp) actions of finitely generated groups on (standard) probability spaces.

- A marked group is a f.g. group Γ with a finite generating set S_Γ, symmetric, not containing the identity element.

- The Cayley graph $\mathrm{Cay}(\Gamma, S_\Gamma)$ of the marked group (Γ, S_Γ) is the simple graph with
  - vertices Γ,
  - edges $\gamma \leftrightarrow \gamma s$, $\gamma \in \Gamma$, $s \in S_\Gamma$.

- The Schreier graphing $\mathcal{G}(\Gamma \curvearrowright (X,\mu), S_\Gamma)$ associated to a pmp action $\Gamma \curvearrowright (X,\mu)$ of a marked group (Γ, S_Γ) is the graph with
  - vertices X
  - edges $x \leftrightarrow s \cdot x$, $x \in X$, $s \in S_\Gamma$

General question

> What kind of properties of the (marked) group and of its actions are intrinsic properties of its Schreier graphings?

Definition

Two pmp actions $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ of two marked groups (Γ, S_Γ) and $(\Lambda\ S_\Lambda)$ are isometric orbit equivalent if the Schreier graphings

$$\mathcal{G}(\Gamma \curvearrowright (X,\mu), S_\Gamma) \text{ and } \mathcal{G}(\Lambda \curvearrowright (Y,\nu), S_\Lambda)$$

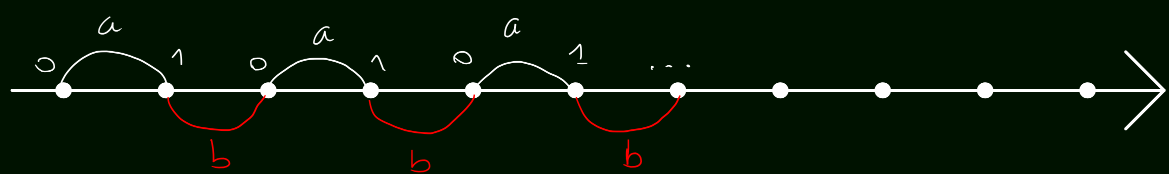
are isomorphic

Example

An isometric OE between pmp free actions of $(\mathbb{Z}, \{\pm 1\})$ and $D_\infty = \langle a, b \mid a^2 = b^2 = 1 \rangle$.

Let $\mathbb{Z} \overset{T}{\curvearrowright} (X,\mu)$ be a pmp free action. Assume that $\exists\ X = X_0 \sqcup X_1$ measurable s.t.

$$\begin{cases} T(X_0) = X_1 \\ T(X_1) = X_0 \end{cases}$$



Two theorems

Theorem A:

$\Gamma = \langle S_\Gamma \rangle$, $\Lambda = \langle S_\Lambda \rangle$ marked groups such that $\mathrm{Cay}(\Gamma, S_\Gamma) \simeq \mathrm{Cay}(\Lambda, S_\Lambda) \simeq G$.

Assume $\mathrm{Aut}(G)$ is discrete.

If two pmp free ergodic actions $(\Gamma, S_\Gamma) \curvearrowright (X, \mu)$ and $(\Lambda, S_\Lambda) \curvearrowright (Y, \nu)$ are isometric OE, then they are virtually conjugate: $\exists\, \Gamma_1 \leq \Gamma$, $\Lambda_1 \leq \Lambda$ isomorphic and of finite index, $X_1 \subseteq X$ a Γ_1-inv. set of >0 measure, $Y_1 \subseteq Y$ a Λ_1-inv. set of >0 measure such that $\Gamma_1 \curvearrowright (X_1, \mu_{X_1})$ and $\Lambda_1 \curvearrowright (Y_1, \nu_{Y_1})$ are conjugate.

A pmp free action $(\Gamma, S_\Gamma) \curvearrowright (X, \mu)$ of a marked group (Γ, S_Γ) is isometric OE rigid if any free pmp action isometric OE to it is in fact virtually conjugate to it.

Two theorems

Theorem B :

$\mathbb{F}_d$ free group on d generators, $S_d = \{a_1, \dots, a_d\}$ a free generating set.
There exists (at least countably many) pmp free ergodic actions of $(\mathbb{F}_d, S_d)$ that are isometric OE non-rigid.

As a corollary of the proof, we get

Proposition : being mixing is not invariant under isometric OE

Part A:

- **A "factor property" for isometric OE:**

(Γ, S_Γ) and (Λ, S_Λ) marked groups with $\mathrm{Cay}(\Gamma, S_\Gamma) \simeq \mathrm{Cay}(\Lambda, S_\Lambda) \simeq \mathcal{G}$.

$\mathrm{Aut}(\mathcal{G})$ the group of graph automorphisms of $\mathcal{G}$ (we have $\Gamma, \Lambda \leq \mathrm{Aut}(\mathcal{G})$).

Let $(\Gamma, S_\Gamma) \curvearrowright (X, \mu)$ and $(\Lambda, S_\Lambda) \curvearrowright (Y, \nu)$ be two pmp free actions that are isometric OE. WLOG, $(X, \mu) = (Y, \nu)$ and the Schreier graphings are the same.

Let $\mathcal{R} = \{(x, y) \in X \times X \mid x \,\&\, y$ are in the same c.c. of the Schreier graphing$\}$

Lemma: There exists a measurable map $\Phi : \mathcal{R} \to \mathrm{Aut}(\mathcal{G})$ s.t.

$\forall \gamma \in \Gamma, \forall \lambda \in \Lambda$, $\forall (x, y) \in \mathcal{R}$: $\quad \Phi(\gamma \cdot x, \lambda \cdot y) = \gamma \, \Phi(x, y) \, \lambda^{-1}$

Part A:

Theorem A:

$\Gamma = \langle S_\Gamma \rangle$, $\Lambda = \langle S_\Lambda \rangle$ marked groups such that $Cay(\Gamma, S_\Gamma) \simeq Cay(\Lambda, S_\Lambda) \simeq G$.

Assume $Aut(G)$ is discrete.

If two pmp free ergodic actions $(\Gamma, S_\Gamma) \curvearrowright (X,\mu)$ and $(\Lambda, S_\Lambda) \curvearrowright (Y,\nu)$ are isometric OE, then they are virtually conjugate

Sketch of proof: We have a measurable map

$$\Phi : \mathcal{R} \to Aut(G)$$

which is $\Gamma \times \Lambda$ equivariant ($\Phi(\gamma \cdot x, \lambda \cdot y) = \gamma \Phi(x,y) \lambda^{-1}$). Then use a lemma due to Furman that says that if such a Φ (taking its its values in a discrete space) exists, then the actions are virtually conj.

Part B:

Theorem B:

$\mathbb{F}_d$ free group on d generators, $S_d = \{a_1, \dots, a_d\}$ a free generating set.
There exists (at least countably many) pmp free ergodic actions of $(\mathbb{F}_d, S_d)$ that are isometric OE non-rigid.

For each $2d$-regular finite graph, we will build two pmp free actions of $\mathbb{F}_d$ that are isometric OE but (under additional assumption on the finite graph) not virtually conjugate.

Part 3:

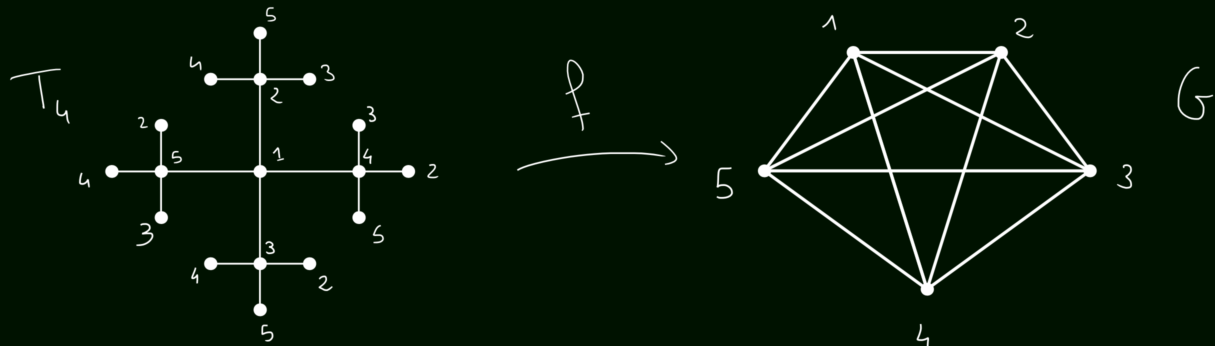
A sketch of proof for $\mathbb{F}_2 = \langle a, b \rangle$:

Let $T_4 = \mathrm{Cay}(\mathbb{F}_2, \{a^{\pm 1}, b^{\pm 1}\})$ 4-regular tree, Fix a 4-regular finite (simple) graph G.

Let $\mathrm{Cov}(T_4, G)$ be the set of covering maps from T_4 to G, i.e. maps $f : V(T_4) \to V(G)$ s.t.

- if $u \leftrightarrow v$ in T_4, then $f(u) \leftrightarrow f(v)$ in $V(G)$,
- for all $u \in V(T_4)$, $f : N_1(u) \to N_1(f(u))$ is a bijection.

Example :



Part 3:

$\mathbb{F}_2$ acts on $\mathrm{Cov}(T_4, G)$ by $\gamma \cdot f = f \circ \gamma^{-1}$.

Let μ be the "uniform" probability measure on $\mathrm{Cov}(T_4, G)$:

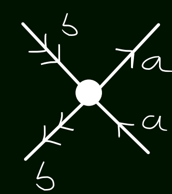
fix a root $\rho \in V(T_4)$, and define a partial covering map on the 1-nbh of ρ uniformly at random. Then continue moving radially outwards through the vertices of T_4, extending the partial covering one step further, doing so by choosing uniformly at random among the possible extensions of the partial covering, indep. at each vertex.

Then $\mathbb{F}_2 \overset{T_G}{\curvearrowright} (\mathrm{Cov}(T_4, G), \mu)$ is a pmp free action.

Part B:

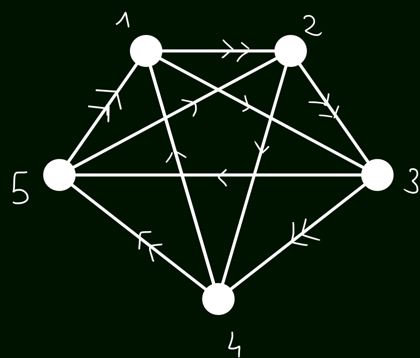
Let's build a pmp free action of $\mathbb{F}_2$ isometric OE to T_G but not virtually conjugate to it.

Fix a schreierisation of the finite graph G: each edge is oriented + labeled by a or b s.t. the 1-nbhd of each vertex looks like



Example:

G

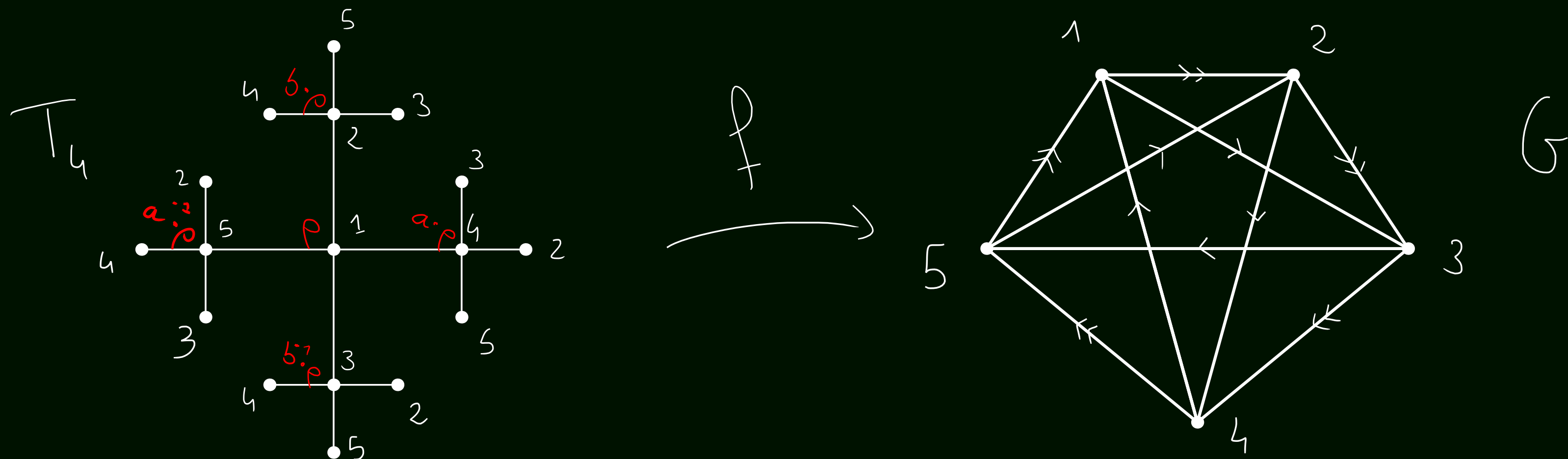


Part 3:

$\rho \in V(T_4)$ the root. Define the action $\mathbb{F}_2 \curvearrowright^{S_G} (\mathrm{Col}(T_4), \mu)$ by:

- <u>How a acts:</u> Look at the 1-nbhd of ρ. Find the unique $s \in \{a^{\pm 1}, b^{\pm 1}\}$ s.t. the edge $f(\rho) \xrightarrow{a} f(s^{-1} \cdot \rho)$ in $V(G)$. Define $\boxed{(S_G)^a f = (T_G)^s f}$.
- <u>How b acts:</u> Look at the 1-nbhd of ρ. Find the unique $s \in \{a^{\pm 1}, b^{\pm 1}\}$ s.t. the edge $f(\rho) \xrightarrow{b} f(s^{-1} \cdot \rho)$ in $V(G)$. Define $\boxed{(S_G)^b f = (T_G)^s f}$.

<u>Example:</u>



Part B:

Why are the actions T_G and S_G not (virtually) conjugate ?

Observation. fix a vertex $v_0 \in V(G)$. Then the set

$$\{ f \in \mathrm{Cov}(T_4, G) \mid f(\rho) = v_0 \} \text{ is } S_G\text{-invariant under } \pi_1(G, v_0) \leq \mathbb{F}_2 .$$

Theorem (J.): The action $\mathbb{F}_2 \overset{T_G}{\curvearrowright} (\mathrm{Cov}(T_4), \mu)$ is mixing iff the finite graph G is non-bipartite.

Thus, when G is non-bipartite, T_G and S_G are not conjugate. A more involved argument implies that they are not virtually conjugate.

Finally, the f-invariant is used to distinguish the actions when G varies.

Part 3:

__Question :__ Are there pmp free actions of $(\mathbb{F}_d, S_d)$ that are isometric or rigid? for instance what about the Bernoulli shift $\mathbb{F}_2 \curvearrowright ([0,1], Leb)^{\mathbb{F}_2}$?

Thank you!