

Topology Review Part 1

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Exercise 1. Derive the exact homology sequence of a pair (X, A) .

Solution 1. We begin with a sequence

$$H_q(A) \xrightarrow{H_q(i)} H_q(X) \xrightarrow{H_q(j)} H_q(X, A)$$

where both maps are homomorphisms induced by inclusion. From this we derive a long exact sequence via a connecting homomorphism

$$\partial_q : H_q(X, A) \rightarrow H_{q-1}(A)$$

defined by

$$\bar{z} \mapsto \overline{\partial z},$$

where ∂ here is the boundary map on chains. It needs to be shown that ∂_q is well-defined, so let z, z' be relative q -cycles representing the same relative homology class \bar{z} . Then $z - z' = w + \partial z''$, where w is a q -chain on A and z'' is a $(q + 1)$ -chain on X . Therefore,

$$\partial z - \partial z' = \partial w + \partial \partial z'' = \partial w,$$

and so ∂z and $\partial z'$ differ by a boundary; $\overline{\partial z} = \overline{\partial z'}$.

Our long exact sequence, then, is

$$\cdots \longrightarrow H_q(A) \xrightarrow{H_q(i)} H_q(X) \xrightarrow{H_q(j)} H_q(X, A) \xrightarrow{\partial_q} H_{q-1}(A) \longrightarrow \cdots .$$

I will show that this sequence is exact by showing that the image of each map is the kernel of the subsequent map.

(i) $\text{im } H_q(i) = \ker H_q(j)$:

$$ji = 0, \text{ so } H_q(j)H_q(i) = 0 \text{ and } \text{im } H_q(i) \subseteq \ker H_q(j).$$

Now let $x \in Z_q(X)$ with $\bar{x} \in \ker H_q(j)$. Then $j(x)$ is a boundary, so $j(x) = \partial z$ for some $z \in S_{q+1}(X, A)$. Since j is surjective, there exists $x' \in S_{q+1}(X)$ with $j(x') = z$, so

$$\begin{aligned} j(x - \partial x') &= j(x) - j(\partial x') \\ &= j(x) - \partial j(x') \\ &= j(x) - \partial z \\ &= j(x) - j(x) \\ &= 0. \end{aligned}$$

Thus, by exactness, $x - \partial x' \in \text{im } i$ and so there exists $w \in S_q(A)$ with $i(w) = x - \partial x'$. Now $i(\partial a) = \partial i(a) = \partial(x - \partial x') = \partial x = 0$, for x is a cycle, and since i is injective $\partial a = 0$: a is a cycle. Therefore, $H_q(i)(\bar{a}) = x - \partial x' = \bar{x}$ and so $\ker H_q(j) \subseteq \text{im } H_q(i)$.

(ii) $\text{im } H_q(j) = \ker \partial_q$:

Let $x \in Z_q(X)$. Then $\partial x = 0$, so that $\partial_q H_q(j)(\bar{x}) = \overline{\partial x} = 0$, and $\text{im } H_q(j) \subseteq \ker \partial_q$. Now let $y \in Z_q(X, A)$ with $\partial_q \bar{y} = 0$. Then $\overline{\partial y} = 0$, so that ∂y is a q -boundary on A . Thus, there exists a q -chain y' with $\partial y' = \partial y$. Therefore, $y - y' \in Z_{q-1}(X)$ satisfying $H_q(j)(\overline{y - y'}) = \bar{y}$ and $\ker \partial_q \subseteq \text{im } H_q(j)$.

(iii) $\text{im } \partial_q = \ker H_{q-1}(i)$:

Let $\bar{x} \in H_q(X, A)$. Then $\partial_q \bar{x} = \overline{\partial x} = 0$, for $\partial x \in B_{q-1}(X)$. Thus, $\text{im } \partial_q \subseteq \ker H_{q-1}(i)$. Now let $w \in Z_{q-1}(A)$ with $H_{q-1}(i)(\bar{w}) = 0$. Then $i(w) = \partial x$ for some $x \in S_q(X)$. Now $\partial j(x) = j(\partial x) = ji(x) = 0$ by exactness of the chain maps, so that $j(x)$ is a cycle. Thus, $\partial_q j(x) = \overline{\partial j(x)} = \bar{w}$, so $\ker H_{q-1}(i) \subseteq \text{im } \partial_q$.

This verifies that the above sequence is exact.

Exercise 2 (Five Lemma). Given a diagram of R -modules and homomorphisms with all rectangles commutative

$$\begin{array}{ccccccccc}
 A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\
 \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \delta \downarrow & & \varepsilon \downarrow \\
 B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5
 \end{array}$$

such that the rows are exact (at the inner joints) and the four outer homomorphisms $\alpha, \beta, \delta, \varepsilon$ are isomorphisms, then γ is an isomorphism.

Proof. First I will show that γ is injective, so let $a \in \ker \gamma$. Then $\delta f_3(a) = 0$ by commutativity, and since δ is an isomorphism, $f_3(a) = 0$. By exactness, $a \in \text{im } f_2$, so let $a' \in A_2$ with $f_2(a') = a$. $\gamma f_2(a') = 0$, so by commutativity $g_2 \beta(a') = 0$. By exactness, then, there exists $b \in B_1$ with $g_1(b) = \beta(a')$. Since α is an isomorphism, $b = \alpha(a'')$ for some $a'' \in A_1$. Now by commutativity $\beta f_1(a'') = g_1 \alpha(a'') = g_1(b) = \beta(a')$; thus, $\beta(a' - f_1(a'')) = 0$. But β is an isomorphism, so $a' = f_1(a'')$, so that by exactness $a = f_2 f_1(a'') = 0$. Therefore, $\ker \gamma$ contains only 0, and γ is injective.

Now I will show that γ is surjective, so let $b \in B_3$. Then since δ is an isomorphism, there exists $a \in A_4$ with $\delta(a) = g_3(b)$. Now $g_4 g_3(b) = 0$, so by commutativity $\varepsilon f_4(a) = 0$; but ε is an isomorphism, so $f_4(a) = 0$. Therefore, by exactness there exists $a' \in A_3$ with $f_3(a') = a$. By commutativity, $\delta f_3(a') = g_3 \gamma(a')$, but $\delta f_3(a') = \delta(a) = g_3(b)$, so $g_3(b - \gamma(a')) = 0$. By exactness there exists $b' \in B_2$ with $g_2(b') = b - \gamma(a')$. β is an isomorphism, so there exists $a'' \in A_2$ with $\beta(a'') = b'$. Now $g_2 \beta(a'') = b - \gamma(a')$. But $g_2 \beta(a'') = \gamma f_2(a'')$ by commutativity, so $\gamma f_2(a'') + \gamma(a') = b$, and $\gamma(f_2(a'') + a') = b$. Therefore, γ is surjective. \square

Exercise 3. If $A \subset X$ is a retract then the long exact homology sequence of the pair (X, A) yields short exact sequences

$$0 \longrightarrow H_q(A) \xrightarrow{H_q(i)} H_q(X) \longrightarrow H_q(X, A) \longrightarrow 0$$

which split via $H_q(r) : H_q(X) \rightarrow H_q(A)$ for all $q \geq 0$. In particular $H_q(X)$ is isomorphic to the direct sum $H_q(A) \oplus H_q(X, A)$.

Proof. Let $r : X \rightarrow A$ be a retraction map; then $ri = id_A$, so $H_q(r)$ and $H_q(i)$ compose to give the identity on $H_q(A)$. Thus $H_q(i)$ must be injective and $H_q(r)$ surjective; because $H_q(i)$ is injective, it has trivial kernel and we can precede $H_q(A)$ by 0 in the exact sequence. Thus, $\text{im } \partial_q = 0$, and so $\ker \partial_q = H_q(X, A)$: $H_q(j)$ is surjective. Therefore, we can follow $H_q(X, A)$ by 0 in the exact sequence, yielding a short exact sequence. Note that this implies that the homology maps of a pair (X, A) will all be either injective or surjective if A is a retract of X . \square