5. THE MAYER-VIETORIS SEQUENCE

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Abstract. We summarize basic facets of the Mayer-Vietoris Sequence (MVS) and present examples and problems that stress its utility.

1. Introduction

To set up the MVS one begins with a triad of spaces (X, X_1, X_2) where the X_i are subspaces of $X = X_1 \cup X_2$. We then have inclusion maps

$$k_1 \;:\; (X_2, X_1 \cap X_2) \longrightarrow (X, X_1) \tag{1.1}$$

$$k_2 \;:\; (X_1, X_1 \cap X_2) \longrightarrow (X, X_2) \tag{1.2}$$

If k_i induces an isomorphism, or in other words if $X_i - X_1 \cap X_2$ can be excised from (X, X_i), then the triad is **exact**.

Note that if the X_i are open then (X, X_1, X_2) is exact. Indeed if we let $A = X_1$, $U = X_1 - X_1 \cap X_2$, then $X - U = (X_1 \cup X_2) - (X_1 - X_1 \cap X_2) = X_2$. Since we assume X_2 is open its complement in X is closed, i.e. U is closed. But since A is open $\overset{\circ}{A} = A$ and since $U \subset A$ the excision theorem applies. Therefore $H_q(k_1)$ is an isomorphism. Similarly $H_q(k_2)$ is also an isomorphism.

The point of the MVS is to compute the homology of a given space by decomposing it into open sets with nonempty intersection, and then using these isomorphisms to obtain a long exact sequence.

If (X, X_1, X_2) is an exact triad we use the Barret-Whitehead Lemma to obtain the MVS:

$$\xrightarrow{h_{q+1}} H_q(X_1 \cap X_2) \xrightarrow{f_q} H_q(X_1) \oplus H_q(X_2) \xrightarrow{g_q} H_q(X) \xrightarrow{h_q} H_{q-1}(X_1 \cap X_2) \xrightarrow{f_{q-1}} \tag{1.3}$$

The MVS is useless if we don't know what the homomorphisms are:

Let $i : X_1 \cap X_2 \hookrightarrow X_1$ and $j : X_1 \cap X_2 \hookrightarrow X_2$ be inclusions. Then

$$\begin{aligned} f_q &= H_q(i) \oplus H_q(j) \\ g_q &= H_q(i) - H_q(j) \end{aligned}$$

Note that g_q is defined using subtraction to ensure exactness at $H_q(X_1) \oplus H_q(X_2)$:

$$\overline{z} \xmapsto{f_q} (\overline{z}, \overline{z}) \overset{g_q}{\mapsto} \overline{z} - \overline{z} = 0$$

The map h_q is defined using the snake lemma. It is the composition of the following three functions:

$$H_q(X, \varnothing) \xrightarrow{H_q(l)} H_q(X, X_2) \xrightarrow{k_2^{-1}} H_q(X_1, X_1 \cap X_2) \xrightarrow{\partial_q} H_{q-1}(X_1 \cap X_2) \tag{1.4}$$

where $l : (X, \varnothing) \longrightarrow (X, X_2)$ is the inclusion, k_2 is the isomorphism defined above, and ∂_q is the connecting homomorphism.

2. Applications

Example 1. *We will compute the homology of the torus X using the MVS. We decompose X as shown below. Since we chose X_1 and X_2 open the triad (X, X_1, X_2) is exact. We can write down the associated MVS (1.3).*
Note that for $q \geq 3$

$$H_q(X_1) \oplus H_q(X_2) = H_{q-1}(X_1 \cap X_2) = H_{q-1}(X_1) \oplus H_{q-1}(X_2) = 0$$

so from the MVS we have

$$0 \longrightarrow H_q(X) \longrightarrow 0 \longrightarrow 0$$

which of course implies that $H_q(X) = 0$. Now consider the part of the MVS:

$$H_2(X_1) \oplus H_2(X_2) \xrightarrow{g_2} H_2(X) \xrightarrow{h_2} H_1(X_1 \cap X_2) \xrightarrow{f_1} H_1(X_1) \oplus H_1(X_2)$$

Since $H_2(X_1) \oplus H_2(X_2) = 0$ this becomes

$$0 \xrightarrow{g_2} H_2(X) \xrightarrow{h_2} H_1(X_1 \cap X_2) \xrightarrow{f_1} H_1(X_1) \oplus H_1(X_2)$$

Therefore h_2 is 1-1 which implies that

$$H_2(X) \cong \operatorname{Im} h_2 = \ker f_1$$

So now we are interested in $\ker f_1$. Recall that $f_1 = H_1(i) \oplus H_1(j)$. Since $H_1(X_2) = \{0\}$ certainly $H_1(j) \equiv 0$. What about $H_1(i)$? It is also the zero map. Indeed let α be a loop in $X_1 \cap X_2$ whose homology class generates $H_1(X_1 \cap X_2)$. The image of α under i is a loop in X_1 that is homotopically equivalent to the loop $a - b - a + b = 0$. Since loops which are homotopically equivalent induce the same *class in homology, we conclude that*

$$H_1(i)(\overline{\alpha}) = 0$$

Hence $H_1(i) \equiv 0$ *as well. Therefore* $f_1 \equiv 0$ *and so*

$$H_2(X) = \ker f_1 = H_1(X_1 \cap X_2) \cong \mathbb{Z}$$

Before going on to compute we note that when $X_1 \cap X_2 \neq \emptyset$ *then the MVS can be terminated with reduced homology:*

$$H_1(X) \longrightarrow H_0^{\#}(X_1 \cap X_2) \longrightarrow H_0^{\#}(X_1) \oplus H_0^{\#}(X_2) \longrightarrow H_0^{\#}(X) \longrightarrow 0$$

Using this we can write

$$H_1(X_1 \cap X_2) \xrightarrow{f_1} H_1(X_1) \oplus H_1(X_2) \xrightarrow{g_1} H_1(X) \xrightarrow{h_0^{\#}} H_0^{\#}(X_1 \cap X_2) \tag{2.1}$$

We've already shown that $\operatorname{Im} f_1 = \{0\}$, *and since* $H_0^{\#}(X_1 \cap X_2) = 0$ *(2.1) can be rewritten as*

$$0 \longrightarrow H_1(X_1) \oplus H_1(X_2) \xrightarrow{g_1} H_1(X) \longrightarrow 0$$

Therefore

$$H_1(X) \cong H_1(X_1) \oplus H_1(X_2) \cong (\mathbb{Z} \oplus \mathbb{Z}) \oplus \{0\} = \mathbb{Z}^2$$

And of course since X *is path connected* $H_0(X) = \mathbb{Z}$.

Example 2. *Now let* X *be the double torus decomposed as shown below. Just as in the case of the torus,* $H_q(X) = 0$ *for* $q \geq 3$. *To compute* $H_2(X)$ *we focus on:*

$$0 \longrightarrow H_2(X) \xrightarrow{h_2} H_1(X_1 \cap X_2) \xrightarrow{f_1} H_1(X_1) \oplus H_1(X_2)$$

h_2 *is 1-1 so* $H_2(X) \cong \operatorname{Im} h_2 = \ker f_1$. *So the question again is, how does* f_1 *behave? For the same reason that* $H_1(i)$ *was the zero map in example 1, in this case* $H_1(i)$ *and* $H_1(j)$ *are the zero maps also. Therefore* f_1 *is the zero map. Hence* $\ker f_1 = H_1(X_1 \cap X_2) \cong \mathbb{Z}$ *so*

$$H_2(X) \cong \mathbb{Z}$$

To compute $H_1(X)$ *it is again useful to look at reduced homology. Since* f_1 *is the zero map (2.1) becomes:*

$$0 \longrightarrow H_1(X_1) \oplus H_1(X_2) \xrightarrow{g_1} H_1(X) \longrightarrow 0$$

Hence $H_1(X) \cong H_1(X_1) \oplus H_1(X_2) = \mathbb{Z}^4$.

Remark 1. *The same ideas can be used to compute the homology of the* n*-holed torus since* f_1 *will always be the zero map.*

Remark 2. *Note that* f_1 *is not 1-1, although it is induced by a 1-1 map. This answers the first question of #6 on our handout. To construct an onto continuous map such that its induced homology is not onto consider the following map* $g : S^1 \longrightarrow S^1 \vee S^1$*. In this case* $H_1(g)(\overline{\alpha}) = (\overline{\alpha_1}, \overline{\alpha_2})$ *Since* $\overline{\alpha}$ *generates* $H_1(S^1)$ *in*

particular $(\overline{\alpha_1}, 0) \notin \operatorname{Im} H_1(g)$*. This answers the second part of #6 on our handout.*

Problem 1. *(Excercise 17.14; last quarter Eric and Paul wrote up a more general solution to this problem that you might want to check out.) Let* D_2 *be the surface obtained by removing 2 open discs from the unit disk* E^2*. We can draw a figure eight around the two missing discs. Then* D_2 *is homotopy equivalent to* $G = S^1 \vee S^1$ *via the map* $r : D_2 \longrightarrow G$ *that we presently describe in two stages. First radially retract the region of* D_2 *lying outside* G *to the boundary of* G *. Then retract the points inside* G *to its boundary. The composition of these two maps will be* r*. Note that this is indeed a retraction since* $r \circ i = Id_G$ *and* $i \circ r \simeq Id_{D_2}$*. Now let* M_2 *be the surface obtained by identifying two copies of* D_2 *,* D_2 *and* D_2'*, along their boundaries. We will use the MVS to show directly that* $H_2(M_2) \cong \mathbb{Z}$ *and* $H_1(M_2) \cong \mathbb{Z} \oplus \mathbb{Z}$*.*

Since E^2 *is normal we can find disjoint neighborhoods around the missing discs and the boundary. In* D_2 *let these neighborhoods be* H_1*,* H_2*, and* J*, where* J *is the open neighborhood containing the boundary of* E^2*. Similarly we have* H_1'*,* H_2'*, and* J' *in* D_2'*.*

Let $X_1 = D_2 \cup H_1' \cup H_2' \cup J'$ *and let* $X_2 = D_2' \cup H_1 \cup H_2 \cup J$*. Then*

$$\begin{aligned} X_1 &\simeq S^1 \vee S^1 \simeq X_2 \\ X_1 \cap X_2 &\simeq S^1 \sqcup S^1 \sqcup S^1 \\ X_1 \cup X_2 &= M_2 \end{aligned}$$

Since the X_i are open we can write down the associated MVS of the triad (M_2, X_1, X_2):

$$\begin{aligned} H_2(X_1) \oplus H_2(X_2) &\longrightarrow H_2(M_2) \xrightarrow{h_2} H_1(X_1 \cap X_2) \xrightarrow{f_1} H_1(X_1) \oplus H_1(X_2) \\ &\longrightarrow H_1(M_2) \xrightarrow{h_1} H_0^{\#}(X_1 \cap X_2) \xrightarrow{f_0} H_0^{\#}(X_1) \oplus H_0^{\#}(X_2) \end{aligned}$$

Using what we already know this becomes:

$$(2.2)\ \ 0 \longrightarrow H_2(M_2) \xrightarrow{h_2} \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{f_1} \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{g_1} H_1(M_2) \xrightarrow{h_1} \mathbb{Z}\oplus\mathbb{Z} \longrightarrow 0$$

Let the generators of $H_1(X_1 \cap X_2)$ be a_1, a_2, and a_3, where a_1 corresponds to the outer boundary. Let b_1, b_2 and b_1', b_2' be generators of $H_1(X_1)$ and $H_1(X_2)$ respectively. So we can write down:

$$f_1 : \mathbb{Z}\langle a_1\rangle \oplus \mathbb{Z}\langle a_2\rangle \oplus \mathbb{Z}\langle a_3\rangle \longrightarrow \mathbb{Z}\langle b_1\rangle \oplus \mathbb{Z}\langle b_2\rangle \oplus \mathbb{Z}\langle b_1'\rangle \oplus \mathbb{Z}\langle b_2'\rangle$$

By orienting all loops in the same direction, we can arrange that

$$\begin{aligned} f_1(a_1) &= (b_1 + b_2, b_1' + b_2') \\ f_1(a_2) &= (b_1, b_1') \\ f_1(a_3) &= (b_2, b_2') \end{aligned}$$

Let's examine f_1 in more detail. The matrix representing f_1 is

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

which has rank 2 and nullity 1. Therefore $\ker f_1 \cong \mathbb{Z}$ and $\operatorname{Im} f_1 \cong \mathbb{Z} \oplus \mathbb{Z}$. Since h_2 is injective we have

$$H_2(M_2) \cong \operatorname{Im} h_2 \cong \ker f_1 \cong \mathbb{Z}$$

From (2.2) we get the short exact sequence:

$$0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_1(M_2) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0$$

where we use that

$$\begin{aligned} \operatorname{Im} f_1 &= \mathbb{Z}\langle b_1 + b_2 + b_1' + b_2'\rangle \oplus \mathbb{Z}\langle b_1 + b_1'\rangle \oplus \mathbb{Z}\langle b_2 + b_2'\rangle \\ &= \mathbb{Z}\langle b_1 + b_1'\rangle \oplus \mathbb{Z}\langle b_2 + b_2'\rangle \end{aligned}$$

and

$$\frac{\mathbb{Z}\langle b_1\rangle \oplus \mathbb{Z}\langle b_2\rangle \oplus \mathbb{Z}\langle b_1'\rangle \oplus \mathbb{Z}\langle b_2'\rangle}{\mathbb{Z}\langle b_1 + b_1'\rangle \oplus \mathbb{Z}\langle b_2 + b_2'\rangle} \cong \mathbb{Z} \oplus \mathbb{Z}$$

Since $\mathbb{Z}^2$ is free this sequence splits and we get that $H_1(M_2) \cong \mathbb{Z}^4$.

Problem 2. *(Excercise 17.17) Given two spaces* $X \subset X'$ *we say* $H_q(X)$ *is an* ***algebraic retract*** *of* $H_q(X')$ *if there exists a map* $\rho : H_q(X') \longrightarrow H_q(X)$ *such that* $\rho \circ H_q(i) = Id_{H_q(X)}$*, where* $i : X \hookrightarrow X'$ *is the inclusion. Note that for topological spaces* $X \subset Y$*, if* X *is a (topological) retract of* Y *then* $H_q(X)$ *is an algebraic retract of* $H_q(Y)$*. We want to show that the converse is not true. Let* X *and* Y *be spaces with* $x_0 \in X$ *and* $y_0 \in Y$*. Assume that* x_0 *and* y_0 *have contractible neighborhoods, so* $X \vee Y$ *is well-defined. We identify* $X \vee Y$ *with* $W = X \times \{y_0\} \cup \{x_0\} \times Y \subset X \times Y$*. We will show* $H_q(X \vee Y)$ *is an algebraic retract of* $H_q(X \times Y)$*.*

Let $j_1 : X \times \{y_0\} \hookrightarrow W$ *and* $j_2 : \{x_0\} \times Y \hookrightarrow W$ *be inclusions. For* $q > 1$ *the MVS gives us an* ***isomorphism*** $\theta : H_q(X \times \{y_0\}) \oplus H_q(\{x_0\} \times Y) \longrightarrow H_q(W)$*, where*

$$\theta = H_q(j_1) - H_q(j_2)$$

Define the map $\gamma : H_q(X \times Y) \longrightarrow H_q(X \times \{y_0\}) \oplus H_q(\{x_0\} \times Y)$ *by*

$$\gamma = \pi_X \oplus \pi_Y$$

where π_X *and* π_Y *are projections. Consider the following diagram:*

$$\begin{tikzcd}
H_q(W) & H_q(X \times Y) \\
H_q(X \times \{y_0\}) \oplus H_q(\{x_0\} \times Y) &
\end{tikzcd}$$

Note that $\pi_X \circ i \circ j_1 = Id_{X \times \{y_0\}}$ *and* $\pi_Y \circ i \circ j_2 = Id_{\{x_0\} \times Y}$*. Now let* $(z, 0) \in H_q(X \times \{y_0\}) \oplus H_q(\{x_0\} \times Y)$*. What is* $\gamma \circ H_q(i) \circ \theta((z, 0))$*?*

$$\begin{aligned}
\gamma \circ H_q(i) \circ \theta((z,0)) &= \gamma \circ H_q(i \circ j_1)(z) \\
&= (H_q(\pi_X \circ i \circ j_1)(z), H_q(\pi_Y \circ i \circ j_1)(z)) \\
&= (z, c_{y_0})
\end{aligned}$$

where c_{y_0} *is the homology class in* $H_q(\{x_0\} \times Y)$ *of the constant map at* y_0*, i.e.* $c_{y_0} = 0$*. In other words* $\gamma \circ H_q(i) \circ \theta((z, 0)) = (z, 0)$*. Similarly for* $(0, w) \in H_q(X \times \{y_0\}) \oplus H_q(\{x_0\} \times Y)$ *we have* $\gamma \circ H_q(i) \circ \theta((0, w)) = (0, w)$*. Therefore* $\gamma \circ H_q(i) \circ \theta$ *is the identity on* $H_q(X \times \{y_0\}) \oplus H_q(\{x_0\} \times Y)$*. Then*

$$\begin{aligned}
\gamma \circ H_q(i) &= \theta^{-1} \\
\theta \circ \gamma \circ H_q(i) &= Id_{H_q(W)}
\end{aligned}$$

Letting $\rho = \theta \circ \gamma$*, we have exhibited the desired retract.*

Note finally that $X \vee Y$ *is not necessarily a (topological) retract of* $X \times Y$*. For instance Ross shows in his notes that* $S^1 \vee S^1$ *is not a retract of* $S^1 \times S^1$ *using results about mapping cones.*

Problem 3. *(Excercise 17.18) The join $X * Y$ of two spaces is the space obtained from $X \times I \times Y$ by making identifications $(x, 1, y)\tilde{}(x', 1, y)$ and $(x, 0, y)\tilde{}(x, 0, y')$. I don't know of good ways to picture this except maybe:*

Admittedly this might not be very helpful, but if the spaces are line segments this is the actual shape. Consider now the subspaces

$$\begin{aligned} C_+X \times Y &= \{[x,t,y] : t \geq 1/2\} \\ X \times C_-Y &= \{[x,t,y] : t \leq 1/2\} \end{aligned}$$

The first thing we want to show is that $(X * Y, X \times C_-Y, C_+X \times Y)$ is exact. We have the inclusion map

$$k_1 : (C_+X \times Y, X \times \frac{1}{2} \times Y) \longrightarrow (X * Y, X \times C_-Y)$$

We hope to excise $X \times C_-Y - \ X \times \frac{1}{2} \times Y = \{[x,t,y] : t < \frac{1}{2}\}$. If we let $V = \{[x,t,y] : t < \frac{1}{3}\}$ then V can be excised since its closure its contained in the interior of $X \times C_-Y - \ X \times \frac{1}{2} \times Y$. Whatsmore we have the obvious deformation retraction $r : \{[x,t,y] : t \geq \frac{1}{3}\} \longrightarrow \{[x,t,y] : t \geq \frac{1}{2}\}$. So we have shown the excision and hence $(X * Y, X \times C_-Y, C_+X \times Y)$ is exact.

Of course we write down the associated MVS:

$$H_{q+1}(X*Y) \xrightarrow{h} H_q(X \times \frac{1}{2} \times Y) \xrightarrow{f} H_q(X \times C_-Y) \oplus H_q(C_+X \times Y) \xrightarrow{g} H_q(X*Y)$$

Note that $C_+X = CX$ and $C_-Y = CY$ which both retract onto a point so $X \times C_-Y \simeq_{\phi_1} X$ and $C_+X \times Y \simeq_{\phi_2} Y$. We rewrite the above sequence:

$$H_{q+1}(X * Y) \longrightarrow H_q(X \times Y) \xrightarrow{\hat{f}} H_q(X) \oplus H_q(Y) \xrightarrow{\hat{g}} H_q(X * Y)$$

What does $\tilde{f}$ look like? The homotopy $\phi_1 : X \times C_-Y \longrightarrow X$ sends $[x,t,y] \xmapsto{\phi_1} x$ and the homotopy $\phi_2 : C_+X \times Y \longrightarrow Y$ sends $[x,t,y] \xmapsto{\phi_2} y$. Now let us write $f = H_q(i) \oplus H_q(j)$; i and j are the relevant inclusions. Then

$$\hat{f} = H_q(\phi_1 i) \oplus H_q(\phi_2 j)$$

Notice that $\phi_1 i = \pi_X$ and $\phi_2 j = \pi_Y$. So $\hat{f}$ is just $H_q(\pi_X) \oplus H_q(\pi_Y)$. But from the previous problem we have a map

$$H_q(X) \oplus H_q(Y) \xrightarrow{\Psi} H_q(X \times Y)$$

such that $\hat{f} \circ \Psi = Id$. This implies that $\hat{f}$ is onto. Hence $\hat{g}$ is the zero map and we can write down

$$0 \longrightarrow H_q(X * Y) \longrightarrow H_{q-1}(X \times Y) \xrightarrow[\hat{f}]{\overset{\Psi}{\curvearrowleft}} H_{q-1}(X) \oplus H_{q-1}(Y) \longrightarrow 0$$

which is a split short exact sequence. Therefore

$$H_q(X * Y) \cong \frac{H_{q-1}(X \times Y)}{\operatorname{Im} \Psi}$$

We're almost done. Notice that $H_q(X) \oplus H_q(Y) \cong X \vee Y$. *Therefore* Ψ, *written as* $\Psi : H_q(X \vee Y) \longrightarrow H_q(X \times Y)$, *is precisely the map induced by the inclusion (by the previous problem). So we can write down the short exact sequence for relative homology:*

$$0 \longrightarrow H_{q-1}(X \vee Y) \xrightarrow{\Psi} H_{q-1}(X \times Y) \longrightarrow H_{q-1}(X \times Y, X \vee Y) \longrightarrow 0$$

which implies that

$$\frac{H_{q-1}(X \times Y)}{\operatorname{Im} \Psi} \cong H_{q-1}(X \times Y, X \vee Y)$$

Putting this all together we have

$$H_q(X * Y) \cong H_{q-1}(X \times Y, X \vee Y)$$

If we assume $(X \times Y, X \vee Y)$ *is a collared pair (for most spaces we run into on a daily basis this isn't much of a restriction) then proposition 19.36 gives*

$$H_{q-1}(X \times Y, X \vee Y) \cong H_{q-1}(\frac{X \times Y}{X \vee Y})$$

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