1. THE ADJUNCTION SPACE MAP

This set of notes is all about using cell complexes (complices?) to study the various projective spaces. So to start off, we need all the basic definitions...

Definition 1.1. *Given two spaces X and Y, a subspace $A \subset X$ and a map $f : A \to Y$, the quotient space $X \cup_f Y$ given by the equation*

$$\frac{X \amalg Y}{\{(a,0) \sim f(a)\}}$$

where a ranges over A is called the adjunction space *of X and Y along f.*

1.1. **If the adjoining map is trivial.** If Y consists of a single point, then the adjunction space $X \cup_f Y$ is just the quotient space X/A.

Now that we have the adjunction space map, we can speak of "attaching n-cells", a concept that turns out to be of grandilomentitudinous value:

Definition 1.2. *In the special case that $(X, A) = (E^n, S^{n-1})$, the adjunction space $Z = E^n \cup_f Y$ is referred to as the attachment of an n-cell to Y. That is, Z is obtained from Y by* attaching an n-cell*.*

Definition 1.3. *A* spherical complex *is obtained by taking a finite set of points and attaching finitely many cells. A* cell complex *is a spherical complex that can be built up from the original set of points by attaching the 1-cells, then the 2-cells, and so on.*

1.2. **What's the difference?** The fact that cell complexes (complices?) are constructible in this dimension-increasing order restricts us from attaching spheres along the interior of higher-dimensional spheres. We can't have a line segment glued to the inside of a disk, or a disk intersecting the interior of E^3.

1.3. **Adjoining to a point.** If the space Y is a single point, then $E^n \cup_f Y \cong S^n$.

Proof. In lower dimensions this is fairly obvious – gluing the endpoints of the line segment E^1 creates a circle; taking the boundary of the disk and wrapping it to a point gives us the sphere.

In general, we can see that $E^n \cup_f Y \cong S^n$ by working backwards. Removing a point (say, the south pole s) gives us a contractible surface, $S^n \setminus \{s\}$. Retract $S^n \setminus \{s\}$ to give the upper hemisphere $\{(x_0, \ldots, x_n) \in S^n | x_n \geq 0\}$, which we can continuously project down to get E^n. Noting that the boundary of E^n consisted of the points about s, we get $E^n \cup_f Y \cong S^n$. ☐

1.4. **The double pancake construction for the sphere.** If f is the inclusion map from S^1 into E^2, then $E^2 \cup_f E^2$ is homeomorphic to the sphere S^2. The map f sends the boundary of E^2 to the boundary of the other copy of E^2, gluing the two disks along their boundaries.

1.5. **Graph theory-style graphs.** A graph embedded in R^n can be thought of as a finite attaching of 1-cells to a set of base points.

2. COLLARED SPACES & THE LIKE

We've seen in class that the adjunction map doesn't just get used for sphereses. This was also the direct precursor to introducing collared pairs:

Definition 2.1. *For spaces $A \subset X$, if A is contained properly in an open set B that strong deformation retracts onto A, the pair (X, A) is called a* collared pair*.*

2.1. **Spheres are good examples for many things.** The pair (E^n, S^{n-1}) is collared, since we can surround S^{n-1} by a "shell", which we can parameterize as $\{x \in E^n : |x| > r\}$, where $0 < r < 1$.

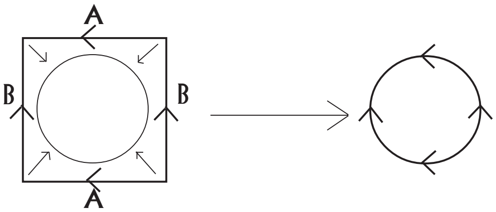
2.2. **A non-spherey example.** An n-dimensional manifold X with a boundary A induces the collared pair (X, A). (An n-dimensional manifold with boundary is a space locally homeomorphic to the half-space $\mathbb{R}^n_+$). For example, the Möbius band, whose boundary is a circle.

The hugetastic result we came up with here was a homeomorphism between collared spaces and adjunction spaces. When passed to homology, this serves as a useful isomorphism:

Proposition 2.2. *Given a collared pair (X, A) and a map $f : A \to Y$ with Y Hausdorff, let $Z = X \cup_f Y$. Then (Z, Y) is a collared pair, and $X - A$ is homeomorphic to $Z - Y$.*

The details of the proof are pretty uninteresting (trust me) and can be found under (19.3) in Harper.

We used this theorem to compute $H_q(S^1 \times S^1, S^1 \vee S^1)$, by strong deformation retracting the boundary of the square to the boundary of E^2:



$S^1 \vee S^1$ has a collaring that retracts onto S^1

3. PROJECTIVE SPACE & CELL COMPLEXES

Since lots of the examples regarding cell complexes (complices? whatever) and attaching maps regard the various projective spaces, it's a good idea to quickly refresh ourselves on those...

Definition 3.1. *The* real projective space, *denoted $\mathbb{P}^n$, is the quotient space*

$$\frac{S^n}{\{x \sim (-x)\}},$$

where x ranges over all of S^n.

Definition 3.2. *Consider in $\mathbb{C}^{n+1}$ and $\mathbb{H}^{n+1}$ respectively the n-spheres $|z| = 1$. Define two points z and z' to be equivalent if $z = cz'$ for some unit complex number (or quaternion). Then the* complex projective space, *denoted $\mathbb{CP}^n$, is equal to the quotient space of the complex n-sphere under $\sim$.*

Similarly, the quaternionic projective space, *denoted $\mathbb{HP}^n$, is equal to the quotient space of the quaternionic n-sphere under $\sim$.*

Before we can prove that the projective spaces are obtained via attaching cells, we need an important lemma:

Lemma 3.3 (The Attaching Lemma)**.** *Suppose $Y \subset Z$ is closed and Z is Hausdorff, and the continuous map $f : E^n \to Z$ maps S^{n-1} onto Y and $\dot{E}^n$ homeomorphically onto $Z - Y$. Then Z is obtained from Y by attaching an n-cell via f (restricted to S^{n-1}).*

Proof. We define $g : E^n \amalg Y \to Z$ by $g|_{E^n} = f$ and $g|_Y = i$, the inclusion map. Since g agrees on the intersection of E^n and Y (namely, since $g(a) = g(f(a))$ for $a \in S^{n-1}$), g is well-defined.

To show that g is an attaching map, we need to show that g is homeomorphic to the map $\frac{E^n \amalg Y}{\{a \sim f(a)\}}$. So we have a diagram that looks like this:

$$\begin{tikzcd}
E^n \amalg Y \arrow[rr, "g"] \arrow[dr, "\Pi"] & & Z \\
& \frac{E^n \amalg Y}{\{a \sim f(a)\}} \arrow[ur, dashed, "h"] &
\end{tikzcd}$$

If we can show there exists a homeomorphism h that makes this diagram commute, then that will show g is an attaching map. The attaching map. First of all, h exists, since a and $f(a)$ are mapped to the same point under g.

If U is open in Z, its inverse image along g is open in $E^n \amalg Y$, and since Π is a quotient map, $\Pi(g^{-1}(U))$ is open in the adjunction space, and since the diagram commutes, we have $h^{-1}(U) = \Pi(g^{-1}(U))$ is open. So h is continuous. Now we only need to show that h is an open map. We will first show that h is closed.

For a closed set V in the adjunction space, $\Pi^{-1}(V)$ is closed in $E^n \amalg Y$ (a closed set in a quotient space is the image of a closed set under Π). But since E^n is compact, $\Pi^{-1}(V) \cap E^n$ is compact, and so $g(\Pi^{-1}(V) \cap E^n)$ is closed. Also, since $\Pi^{-1}(V) \cap Y$ is closed, so is $g(\Pi^{-1}(V) \cap Y)$. Therefore, $g(\Pi^{-1}(V))$ is closed, and by commutivity $h(V) = g(\Pi^{-1}(V))$ is closed as well. But then h is open as well, since h is a one-to-one function. Therefore, h is a homeomomrphism, and we're finished. $\square$

Okay, now that that ugliness is behind us, we're ready to start building some projective spaces. You'll need some glue, some macaroni noodles, and a piece of colored construction paper:

3.1. **Building the projective spaces.** The different projective spaces can be created by attaching cells. Inductively, we can build the real (complex, and quaternionic) projective space $\mathbb{P}^n$ ($\mathbb{CP}^n$, $\mathbb{HP}^n$) by attaching an n-cell ($2n$-cell, $4n$-cell) to $\mathbb{P}^{n-1}$ ($\mathbb{CP}^{n-1}$, $\mathbb{HP}^{n-1}$).

Proof. We will build $\mathbb{P}^n$ off of $\mathbb{P}^{n-1}$. If we let f be the quotient map from S^{n-1} to $\mathbb{P}^{n-1}$

$$\begin{tikzcd}
S^{n-1} \arrow[r, "i"] \arrow[d, "f"'] & E^n \arrow[d, "f'"'] \\
\mathbb{P}^{n-1} \arrow[r, "j"] & \mathbb{P}^n
\end{tikzcd}$$

The map j sends $(x_1, \ldots, x_n)$ to $(0, x_1, \ldots, x_n)$, the map i is inclusion, and f' sends $x = (x_1, \ldots, x_n)$ to $(\sqrt{1 - |x|^2}, x_1, \ldots, x_n)$.

We want to show that f' is one-to-one on the interior of E^n (which we usually denote as $\dot{E}^n$). If $f'(x) = f'(y)$, we need to show that $x = u \cdot y$, where u is a unit. (Of course for the real projective space $u = \pm 1$, but by leaving u as a general unit will hopefully make the other two versions of this proof more obvious).

Suppose then that $f'(x) = f'(y)$, and that $|x| < 1$ (so x is not on S^{n-1}). Then the first coordinate of $f'(x)$ is a positive real number, which forces $u = 1$. But this means $x = y$, so f' restricted to $\dot{E}^n$ is one-to-one, and its image is $\mathbb{P}^n \setminus \mathbb{P}^{n-1}$, since we miss only those equivalence classes with a zero in the first coordinate.

Since f' sends $\dot{E}^n$ homeomorphically onto $\mathbb{P}^n \setminus \mathbb{P}^{n-1}$ and S^{n-1} onto $\mathbb{P}^{n-1}$, f' is an attaching map that adds an n-cell to $\mathbb{P}^n \setminus \mathbb{P}^{n-1}$ to get $\mathbb{P}^n$ (by the attaching lemma).

The proof for the complex and quaternionic projective spaces is exactly the same, save for replacing n with $2n$ and $4n$. In both cases, the unit u still must be 1 in order for $\sqrt{1-|y|^2}$ to be positive real, and so f' again is one-to-one and onto the respective spaces $\mathbb{CP}^n \setminus \mathbb{CP}^{n-1}$ and $\mathbb{HP}^n \setminus \mathbb{HP}^{n-1}$. □

3.2. **The first projective spaces are spheres.** We are going to show the following:

- ♣ $\mathbb{P}^1 \cong S^1$
- ♢ $\mathbb{CP}^1 \cong S^2$
- ♡ $\mathbb{HP}^1 \cong S^4$

Proof. Since we can build these spaces inductively, we immediately have the various projective spaces are attachments of 1-, 2, and 4-cells to the original base point, and so they must all be congruent to the respective n-spheres (by 1.3). □

3.3. **Pictures speak louder than... corollaries?** We can show that $\mathbb{P}^1 \cong S^1$ by noticing that $S^1 \cong [0,2\pi]/\{0,2\pi\}$ (when representing elements of S^1 by their angle), and $\mathbb{P}^1 \cong [0,\pi]/\{0,\pi\}$ (by considering the smallest angle in each equivalence class). Since $[0,2\pi]/\{0,2\pi\} \cong [0,\pi]/\{0,\pi\}$ (the homeomorphism is left as an exercise to the reader), we have $\mathbb{P}^1 \cong S^1$.