Topology Writeup #8

Nate Eldredge

March 11, 2005

Abstract

We construct the lens space in two different ways, examine some properties, and compute its homology. We show that any spherical complex has finitely generated homology. Lastly, we illustrate how to compute the homology of a space obtained by identifying the edges of a polygon.

1 Lens spaces

1.1 Construction as a quotient of S^3 (19.12)

Consider S^3 as the unit sphere in $\mathbb{C}^2$ (i.e. $S^3 = \{(z_0, z_1) \in \mathbb{C}^2 : |z_0|^2 + |z_1|^2 = 1\}$). Then fix relatively prime integers p, q and let $h : S^3 \to S^3$ be defined by

$$h(z_0, z_1) = (e^{2\pi i/p} z_0, e^{2\pi i q/p} z_1).$$

It is clear that h is continuous and has the continuous inverse $h^{-1}(z_0, z_1) = (e^{-2\pi i/p} z_0, e^{-2\pi i q/p} z_1)$, so h is a homeomorphism. Moreover, applying h p times gives the identity, though not for any smaller number of times. Thus we have the group $G = \{id, h, h^2, \dots, h^{p-1}\}$ of homeomorphisms acting on S^3. Since this group is finite, its action must be properly discontinuous. (S^3 is Hausdorff, so given any $x \in S^3$ we can find disjoint neighborhoods $U_0 \ni x$, $U_1 \ni h(x)$, $U_2 \ni h^2(x)$, $\dots$, $U_{p-1} \ni h^{p-1}(x)$. Then $U = \cap_{k=0}^{p-1} h^{-k}(U_k)$ is a neighborhood of x such that $U \cap h^k(U) = \emptyset$ for $k = 0, 1, \dots, p-1$.) We can then take the quotient of S^3 under this action, i.e. identify $x \sim h(x)$. The resulting quotient space is called the **lens space** $L(p, q)$.

As an aside, finding a group of homeomorphisms of order p of S^3 is possible only because 3 is odd. We showed in Exercise (16.11) that any group of homeomorphisms acting on S^{2n} can only have order 1 or 2.

Proposition 1.1. *$L(p, q)$ is a compact connected 3-manifold.*

Proof. It is immediately clear that $L(p, q)$ is compact and connected (since S^3 is, and the quotient map π is continuous). Moreover, $L(p, q)$ is a 3-manifold. If $\pi(x)$ is any point of $L(p, q)$, let U be a neighborhood of $x \in S^3$ as defined above. π is a bijection on U, hence a homeomorphism (quotient maps are continuous and open). Thus, since x has a neighborhood $V \subset U$ which is homeomorphic to $\mathbb{R}^3$, $\pi(V)$ is such a neighborhood of $\pi(x)$. □

1.2 Construction as an adjunction space (21.27)

We can also construct the lens space as an adjunction space, which will make it easier to compute its homology. This construction yields a space homeomorphic to our previous definition of $L(p, q)$, but we shall not prove this.

The idea is to take two copies of the solid torus $E^2 \times S^1$ and glue them together along their boundaries. The gluing map will be a homeomorphism h of $\partial(E^2 \times S^1) = S^1 \times S^1$ which takes a meridian and winds it q times in the meridian direction and p times in the longitude direction. See Figure 1. (We use the letter h to be consistent with the book, but this h is not related to the h in the previous subsection.)

Recall that $H_1(S^1 \times S^1; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$, and one possible basis is obtained from the meridian and longitude cycles m and l. So more precisely, the condition we want is that

$$H_1(h)(m) = qm + pl.$$

It is not immediately obvious that such an h must exist for any given p, q. However, in (21.23–26) it is shown that such an h can be constructed as a composition of "twist" homeomorphisms. Specifically, if A is any 2×2 matrix with integer entries and $\det A = \pm 1$, there will exist an h with $H_1(h) = A$ (with respect to the $\{m, l\}$ basis). In our case, we want $A = \left(\begin{smallmatrix} q & a \\ p & b \end{smallmatrix}\right)$, where we don't care what a, b are. (They will tell us what $H_1(h)(l)$ is, but we aren't going to specify what that should be.) Note that in order to have $\det A = \pm 1$, we need $bq - ap = \pm 1$. This is possible iff (p, q) are relatively prime, which is why we made that requirement.

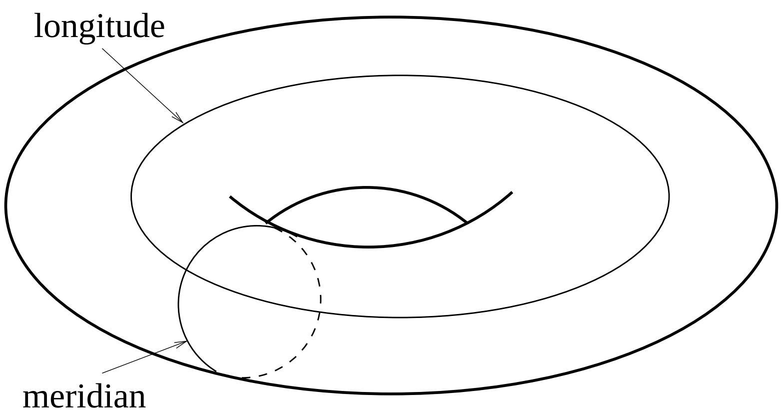

Figure 1: Generating cycles of $H_1(S^1 \times S^1)$.

Thus, let V_1, V_2 be our solid tori. Clearly $(V_1, \partial V_1)$ is a collared pair, and if we view our h as mapping ∂V_1 into $\partial V_2 \subset V_2$, we can form the adjunction space $L(p,q) = V_1 \cup_h V_2$. (Recall this is by definition a quotient of the disjoint union: $V_1 \cup_h V_2 := \frac{V_1 \amalg V_2}{x \sim h(x)}$.) This construction would appear to depend on our specific choice of the map h, but in fact it can be shown that it does not; just so long as $H_1(h)(m) = H_1(h')(m) = qm + pl$, then we will have $V_1 \cup_h V_2 \cong V_1 \cup_{h'} V_2$.

Notice it is clear from this construction that $L(p,q)$ is compact and connected.

Under this definition, we can now compute the homology.

Proposition 1.2. *The integral homology of the lens space $L(p,q)$ is as follows:* [1]

$$H_n(L(p,q);\mathbb{Z}) \cong \begin{cases} 0, & n > 3 \\ \mathbb{Z}, & n = 3 \\ 0, & n = 2 \\ \mathbb{Z}_p, & n = 1 \\ \mathbb{Z}, & n = 0. \end{cases}$$

Proof. For brevity, we write L for $L(p,q)$.

[1] By $\mathbb{Z}_p$, I mean $\mathbb{Z}/p\mathbb{Z}$, the integers mod p, as God intended. We'll have none of that heretical "submodule annihilated by multiplication by p" nonsense here.

Greenberg's (19.15) tells us that for any n, we have the Mayer-Vietoris exact sequence

$$\ldots \to H_n(\partial V_1) \to H_n(V_1) \oplus H_n(V_2) \to H_n(L) \to H_{n-1}(\partial V_1) \to \ldots.$$

Now V_1 and V_2 are solid tori and hence homotopically equivalent to S^1, so that $H_n(V_1) = H_n(V_2) = 0$ for $n \geq 2$. Moreover, $\partial V_1 = S^1 \times S^1$. Thus for $n \geq 3$ we get $H_n(L) \cong H_{n-1}(S^1 \times S^1)$, which is $\mathbb{Z}$ for $n = 3$ and 0 for $n > 3$.

To deal with the case $n = 1, 2$, we must be more specific. It will be more convenient to apply (19.15) with reduced homology; the proof only used the exactness of the homology sequence, so it still works in the reduced case. We have $H_2(V_1) = H_2(V_2) = 0$, and also $H_0^{\#}(\partial V_1) = 0$. So (19.15) gives the exact sequence

$$0 \to H_2(L) \xrightarrow{\phi} H_1(\partial V_1) \xrightarrow{(H_1(j), H_1(h))} H_1(V_1) \oplus H_1(V_2) \xrightarrow{\psi} H_1(L) \to 0$$

where $j : \partial V_1 \to V_1$ is inclusion and $h : \partial V_1 \to V_2$ is as above. [2] ϕ and ψ are as given in the Meyer-Vietoris sequence; for this problem, we don't need to know exactly what they are.

Now $\partial V_1 = S^1 \times S^1$ is a torus, so $H_1(\partial V_1) \cong \mathbb{Z}\langle m\rangle \oplus \mathbb{Z}\langle l\rangle$, where m and l are the "meridian" and "longitude" cycles. And for the solid tori V_1, V_2 we write $H_1(V_1) \cong \mathbb{Z}\langle l_1\rangle$, $H_1(V_2) \cong \mathbb{Z}\langle l_2\rangle$, where l_1, l_2 can be viewed as longitude cycles. Then for $h : \partial V_1 \to V_2$, we have $H_1(h)(m) = pl_2$. (A qm term originally appeared, but we are now mapping into the *solid* torus, where the meridian cycle is trivial.) We do not specify what $H_1(h)(l)$ ought to be; it is some element $xl_2 \in H_1(V_2) = \mathbb{Z}\langle l_2\rangle$. As for the inclusion $j : \partial V_1 \to V_1$, we clearly have $H_1(j)(m) = 0$, $H_1(j)(l) = l_1$. (Longitude is sent to longitude, and again the meridian cycle becomes trivial.) Therefore the matrix of the map $(H_1(j), H_1(h))$ is

$$\begin{pmatrix} 0 & 1 \\ p & x \end{pmatrix}.$$

This is injective, so $\ker(H_1(j), H_1(h)) = 0$. To compute the image, we change the basis on $H_1(V_1) \oplus H_1(V_2)$ and write it as $\mathbb{Z}\langle l_2\rangle \oplus \mathbb{Z}\langle l_1 + xl_2\rangle$. Then we clearly have $\operatorname{im}(H_1(j), H_1(h)) = p\mathbb{Z}\langle l_2\rangle \oplus \mathbb{Z}\langle l_1 + xl_2\rangle \cong p\mathbb{Z} \oplus \mathbb{Z}$.

So by exactness, we have $\operatorname{im}\phi = \ker(H_1(j), H_1(h)) = 0$. ϕ is also injective, so $H_2(L) = 0$. Also, since ψ is onto, by the first isomorphism theorem we

[2] Greenberg and Harper write $(H_1(j) \oplus H_1(h)) \circ \Delta$ instead of my $(H_1(j), H_1(h))$, which seems to me more clear. They both mean the map $\alpha \mapsto (H_1(j)(\alpha), H_1(h)(\alpha))$.

have

$$\begin{aligned}
H_1(L) &\cong \frac{H_1(V_1) \oplus H_1(V_2)}{\ker \psi} \\
&= \frac{H_1(V_1) \oplus H_1(V_2)}{\operatorname{im}(H_1(j), H_1(h))} && \text{by exactness} \\
&\cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{p\mathbb{Z} \oplus \mathbb{Z}} \\
&\cong \mathbb{Z}_p.
\end{aligned}$$

L is connected as we claimed, and so $H_0(L) = \mathbb{Z}$. This completes the proof.

□

Also, we know L is compact and connected, and we showed using the other construction that L is a 3-manifold. Therefore since $H_3(L) = \mathbb{Z}$, Corollary (22.28) yields:

Corollary 1.3. *$L(p, q)$ is orientable for all relatively prime (p, q).*

This is by no means obvious from the construction.

We have also shown that $L(p, q) \neq L(p', q')$ if $p \neq p'$, though we do not know what happens if $p = p'$ but $q \neq q'$. In fact, it has been shown that $L(p, q) \cong L(p, q')$ iff $\pm q' \equiv q^{\pm 1} \pmod p$. Also, there are pairs such as $L(7, 1)$ and $L(7, 2)$ which are homotopically equivalent (and thus have the same homology) but are not homeomorphic.

2 The homology of a spherical complex is finitely generated

Proposition 2.1 (19.20). *Let Z be a spherical complex, and R a Noetherian ring. Then $H_q(Z; R)$ is finitely generated.*

Recall that a spherical complex is obtained by starting with finitely many points (0-cells) and attaching finitely many cells. Also, Noetherian rings (which we shall not define) include $\mathbb{Z}$, $\mathbb{Z}_n$, and any field, so for our purposes we needn't worry much about that condition.

We shall need a couple of algebraic facts, which we won't prove.

Fact 2.2. *If R is a Noetherian ring and M is a finitely generated R-module, then so is any submodule, quotient or homomorphic image of M.*

Fact 2.3. *Suppose M is an R-module and $N \subset M$ is a submodule. If N and M/N are finitely generated, then so is M.*

Now we prove Proposition 2.1.

Proof. By induction on the number of cells attached. Clearly if no cells are attached, Z consists of some finite number k of discrete points. Then $H_0(Z) = R^k$, $H_q(Z) = 0$ for $q > 0$, and these are finitely generated.

Now suppose Y is a cell complex with finitely generated homology and that Z is obtained from Y by attaching an n-cell via a map $f : S^{n-1} \to Y$. We apply Corollary (19.16–18).

For $q \neq n, n-1$ we have $H_q(Z) \cong H_q(Y)$ which is finitely generated.

For $q = n-1$, we have $H_{n-1}(Z) \cong H_{n-1}(Y)/\operatorname{im} H_{n-1}(f)$. Thus $H_{n-1}(Z)$ is a quotient of a finitely generated R-module, hence it is itself finitely generated by Fact 2.2.

For $q = n$, we have the following short exact sequence:

$$0 \to H_n(Y) \xrightarrow{\phi} H_n(Z) \xrightarrow{\psi} \ker H_{n-1}(f) \to 0$$

where the map ϕ is induced by inclusion $Y \to Z$, and ψ comes from the connecting homomorphism. Now ψ is onto, so by the first isomorphism theorem, $H_n(Z)/\ker\psi \cong \ker H_{n-1}(f)$. However, $\ker H_{n-1}(f) \subset H_{n-1}(S^{n-1}) \cong R$ which is finitely generated, hence by Fact 2.2 so is $\ker H_{n-1}(f)$. Also, $H_n(Y)$ is finitely generated by assumption, so by Fact 2.2 so is $\operatorname{im}\phi$, and $\ker\psi \cong \operatorname{im}\phi$ by exactness. Therefore, using Fact 2.3, $H_n(Z)$ is finitely generated. □

3 Homology of a surface obtained by identifying edges of a polygon

A variety of surfaces can be constructed by taking a polygonal disk and gluing its edges together in various ways. In this section, we give an example to illustrate how to compute the homology of such a surface.

Figure 2 shows a 7-gon P with some randomly selected edges identified. To construct our surface X, we view P as a 1-skeleton and attach to it a 2-cell.

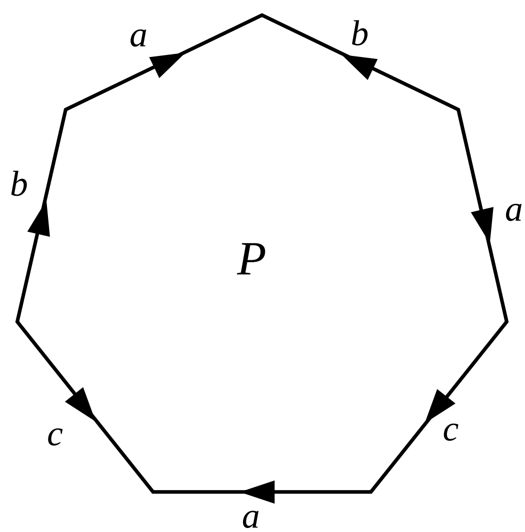

Figure 2: A polygon P with some sides identified.

First of all, a moment's inspection will confirm that all the vertices of P have been identified with one another. Therefore, each of the edges a, b, c are actually loops (since their endpoints are identified) which meet at this common point. P is thus a wedge of three circles: $P = S^1 \vee S^1 \vee S^1$. We can thus write $H_1(P) = \mathbb{Z}\langle a\rangle \oplus \mathbb{Z}\langle b\rangle \oplus \mathbb{Z}\langle c\rangle$. Of course, we also get $H_0(P) = \mathbb{Z}$ and $H_q(P) = 0$ for $q \geq 2$.

We now wish to attach a 2-cell along P. Since P is in fact a quotient of S^1, we effectively let $f : S^1 \to P$ be the quotient map. Then our desired X is the adjunction space $E^2 \cup_f P$. (Observe that X is compact and connected.) Let z be the generator of $H_1(S^1)$ which winds clockwise. Then from the picture (starting at the top), we see that $H_1(f)(z) = -b + a + c + a - c + b + a = 3a$. Specifically, $H_1(f)$ is injective, and $\operatorname{im} H_1(f) = 3\mathbb{Z}\langle a\rangle$.

Now we apply (19.16–19). By (19.16) for $q > 2$ we have $H_q(X) \cong H_q(P) = 0$, and $H_0(X) \cong H_0(P) \cong \mathbb{Z}$. By (19.17) we get

$$H_1(X) \cong \frac{H_1(P)}{\operatorname{im} H_1(f)} = \frac{\mathbb{Z}\langle a\rangle \oplus \mathbb{Z}\langle b\rangle \oplus \mathbb{Z}\langle c\rangle}{3\mathbb{Z}\langle a\rangle} \cong \mathbb{Z}_3 \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

And $\ker H_1(f) = 0$ is free, so (19.19) applies, and we get

$$H_2(X) \cong H_2(P) \oplus \ker H_1(f) = 0 \oplus 0 = 0.$$

In summary, we have shown

$$H_q(X;\mathbb{Z}) = \begin{cases} 0, & q \geq 2 \\ \mathbb{Z}_3 \oplus \mathbb{Z} \oplus \mathbb{Z}, & q = 1 \\ \mathbb{Z}, & q = 0. \end{cases}$$

As an aside, we could use a Seifert-Van Kampen theorem argument to compute $\pi_1(X)$. The resulting group can be written

$$\pi_1(X) = \langle a, b, c \mid b^{-1}acac^{-1}ba = 1 \rangle$$

which is much harder to deal with. However, we now know its abelianization must be $H_1(X) = \mathbb{Z}_3 \oplus \mathbb{Z} \oplus \mathbb{Z}$.

Also, calling X a “surface” is something of a misnomer. A space obtained in this fashion need not be a 2-manifold. In our case, it seems there may be some difficulty with points along the a edge; there are three edges glued together here, and it is not obvious that such a point has a neighborhood homeomorphic to $\mathbb{R}^2$. In fact, it turns out that X is not a manifold; to confirm this, we compute $H_2(X;\mathbb{Z}_2)$.

Analogously to the $\mathbb{Z}$ case, we have $H_2(P;\mathbb{Z}_2) = 0$, $H_1(P;\mathbb{Z}_2) \cong \mathbb{Z}_2\langle a \rangle \oplus \mathbb{Z}_2\langle b \rangle \oplus \mathbb{Z}_2\langle c \rangle$, and $H_1(S^1;\mathbb{Z}_2) \cong \mathbb{Z}_2\langle z \rangle$. Then $H_1(f;\mathbb{Z}_2)(z) = -b + a + c + a - c + b + a = a \neq 0$, so that again $\ker H_1(f;\mathbb{Z}_2) = 0$. This is free, so (19.19) applies and tells us that

$$H_2(X;\mathbb{Z}_2) \cong H_2(P;\mathbb{Z}_2) \oplus \ker H_1(f;\mathbb{Z}_2) = 0.$$

But Corollary (22.30) says that for a compact connected n-manifold M, $H_n(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$, because any manifold is $\mathbb{Z}_2$-orientable. Our X is compact and connected, so it cannot be a manifold.