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290 Review 9

Proposition 1. For a collared pair, the reduced homology of the pair (X,A) is the same as the homology of X/A. $H_q^\sharp(X/A) \cong H_q(X,A)\ \forall q$

Proof. What is the space X/A? It is the space obtained from X by collapsing A to a point. Another way to think of it is as adjoining X to the space Y (= a point p) via the constant map $f : A \to p$. So X/A $\cong$ Z.

From the relative homology sequence of $H_q(Z,Y)$, we know the following sequence is exact:

$H_q^\sharp(Y) \to H_q^\sharp(Z) \to H_q^\sharp(Z,Y) \to H_{q-1}^\sharp(Y)$

But $Y \cong p$, so $H_q^\sharp(Y) = 0\ \forall q$, so substituting $X/A = Z$, the sequence becomes:

$0 \to H_q^\sharp(X/A) \to H_q^\sharp(Z,Y) \to 0$ i.e. $H_q^\sharp(X/A) \cong H_q^\sharp(Z,Y)$. Since Y is non-empty, we know from page 81 that $H_q^\sharp(Z,Y) = H_q(Z,Y)$, so $H_q^\sharp(X/A) \cong H_q(Z,Y)$

Then by 19.14, we know that: $H_q(X,A) \cong H_q(Z,Y)$

Therefore $H_q^\sharp(X/A) \cong H_q(X,A)\ \forall q$

Examples

When we calculated the homology of the torus, we needed to know $H_q(T, S^1 \vee S^1)$, which we now know is equal to $H_q(T/(S^1 \vee S^1))$ should we need it.

In our calculation of the homology of spheres, it is useful in calculating various relative homologies of disks and spheres to instead use the above proposition. $H_q(E^n, S^{n-1}) \cong H_q^\sharp(E^n/S^{n-1})$ by our proposition, but is also $\cong H_{q-1}^\sharp(S^{n-1})$ by 14.3.

When we were computing the homology of $\mathbb{P}^n$, we mapped $H_n(S^n, S^{n-1})$ to $H_n(\mathbb{P}^n, \mathbb{P}^{n-1})$. We now know that $H_n(S^n, S^{n-1}) \cong H_n(S^n/S^{n-1})$, which is $\cong H_n(S^n \vee S^n)$, which is easy to compute using the fact that homology of the wedge is equal to the direct sum of homologies (this is explained in detail in another set of notes).

Exercise 19.42

If $f : S^n \to S^n$ satisfies $f(-x) = f(x)\ \forall x \in S^n$, then deg(f) is even; in particular, if n is even, then deg f = 0.

Let p be the standard projection map from $S^n to P^n$. Define $g([x]) : P^n \to S^n$ by g([x])=f(x). Then f = g(p). Let n be odd. By 19.23 $H_n(g \circ p)(\alpha) = H_n(g)(2\alpha) = 2H_n(g)(\alpha) = H_n(f)(\alpha)$. So $H_n(f)(\alpha)$ is even.

Let n be even. Then by 19.23 $H_n(g \circ p)(\alpha) = 0 = H_n(f)(\alpha)$.

Exercise 19.43

If $f : S^n \to S^n$ has odd degree, then $f(-x) = -f(x)$ for some $x \in S^n$.

Assume not. Then $\forall x \in S^n$, $f(-x) \neq -f(x)$, i.e. $f(-x) + f(x) \neq 0$.

Consider $h(x) = \frac{f(x)+f(-x)}{|f(x)+f(-x)|}$. Since the denominator is never zero, h(x) is continuous. Notice $h : S^n \to S^n$, and $h(-x) = h(x) \forall x$. Then by 19.42, h has even degree.

Examine $F(x,t) : \frac{(1-t)f(x)+t(f(-x))}{|(1-t)f(x)+t(f(-x))|}$

Since $f(-x) + f(x) \neq 0$, $f(x)$ and $f(-x)$ are never connected by a line thru the origin, so neither are $(1-t)f(x)$ and $t(f(-x))$, so $(1-t)f(x) + t(f(-x))$ is never zero. Then $F(x,t)$ is continuous.

$F(x,0) = \frac{f(x)}{|f(x)|} = f(x)$

$F(x, \frac{1}{2}) = \frac{\frac{1}{2}f(x)+\frac{1}{2}f(-x)}{\left|\frac{1}{2}f(x)+\frac{1}{2}f(-x)\right|} = h(x).$

Replacing $t = s/2$, then $s \in [0,1]$ makes F a homotopy between f and h, so f and h have the same degree. But h has even degree, and f has odd degree. Impossible. Then $f(-x) = -f(x)$ for some $x \in S^n$.

Exercise 19.44 Show $\mathbb{P}^k$ is not a retract of $\mathbb{P}^n$ if either n is odd and $n-k$ is even, or n is even and $n-k$ is odd. Could this be done using results only for $R = \mathbb{Z}/2\mathbb{Z}$?

We recall that the homology of $\mathbb{P}^k$ is given by $H_q(\mathbb{P}^k; R) = 0$ if $q > n$,
R_2 if q even such that $1 < q \leq n$,
$R/2$ if q odd such that $1 \leq q \leq n-1$,
R if $q = 0$ and $q = n$ if n is odd

Now, if $\mathbb{P}^n$ retracts onto $\mathbb{P}^k$, then we have composition $r \circ i \simeq \mathrm{id}_{\mathbb{P}^n}$, where r and i are the pertinent retraction and inclusion. Applying the homology functor, $H_k(\cdot; \mathbb{Z})$, to this composition and that our hypothesis requires k to be odd yields

$$H_k(\mathbb{P}^k; \mathbb{Z}) \to H_k(\mathbb{P}^n; \mathbb{Z}) \to H_k(\mathbb{P}^k; \mathbb{Z})$$

$$\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}$$

where the composition should be the identity. Clearly this is absurd, hence our result. We note that if we use only results with $R = \mathbb{Z}/2\mathbb{Z}$ the above argument falls apart, as all of the above homologies are isomorphic to R. ■

We therefore know that if k is odd, k-dimensional projective space is not a retract of n-dimensional projective space. We know an additional result, that $n-1$-dimensional projective space is also not a retract, due to the fact that the covering map from the n-sphere to projective space is not nullhomotopic and then using 21.17 (another set of notes explains this in more detail).

Constructing Spaces with a given Homology

Given any finitely generated abelian group G, we can construct a connected space X with j-th homology group isomorphic to G and all other homology groups of X zero except in degree 0.

The group G can be written as a direct sum of $\mathbb{Z}$'s and $\mathbb{Z}_n$'s, e.g.

$G = \mathbb{Z}^a \oplus \mathbb{Z}_{n_1}^{a_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}^{a_k}$

For each different n_i, construct a_i circles. Attach 2-cells to these circles, using the map that takes S^1 to S^1 by wrapping around the circle counterclockwise n_i times. The image of this attaching map is $n_i\mathbb{Z}$, the kernel of this map is 0. Then using 19.16-19.18, we find the reduced homology is zero in all dimensions except 1. Each of these spaces has first degree homology $\mathbb{Z}_{n_i}^{a_i}$. (We had a homework problem which showed this in the case $n_i = 3$, but it is also true in general.) Also construct a copies of S^1. Then one-point union (wedge) the spaces and the circles by their base points. Call this space Y. $H_1(Y) = G = \mathbb{Z}^a \oplus \mathbb{Z}_{n_1}^{a_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}^{a_k}$ by construction, since the homology of the wedge of spaces is the direct sum of the homologies (this is discussed in another section of notes). Suspend this space j-1 times, and the resulting space is the one desired, since suspension raises the non-zero homologies by 1.

H_1 is not as sensitive as Π_1.

Example: Two spaces that have isomorphic homology groups but different homotopy types.

Let $X = T \vee T$, and let $Y = S^1 \vee S^1 \vee S^1 \vee S^1 \vee S^2 \vee S^2$, all joined at the same basepoint.

Then $H_q(X) = R$ for $q = 0$ or $q \geq 3$, $H_q(X) = R \oplus R$ for $q = 2$, and $H_q(X) = R \oplus R \oplus R \oplus R$ for $q = 1$. Likewise for $H_q(Y)$.

$\Pi_1(T) = Z \times Z$, so $\Pi_1(X) = Z \times Z * Z \times Z$. However, using Van Kampen's Theorem, $\Pi_1(Y) = Z * Z * Z * Z$, so X and Y have the same homology, but different homotopy type.

Another example which does not appeal to the fundamental group is the wedge of CP^{n-1} and S^{2n}, which has the same homology as CP^n. However, they are not homotopy equivalent because they have different cohomology rings. We will show this in class later. A corollary of this is that the 3rd homotopy group of the 2-sphere is nontrivial.