Topology Writeup # 10 & 11

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1. Euler Characteristic

We can define Euler characteristic directly with homology over any ring R a PID.

Definition 1.1. *For a topological space* X*, the rank* $\operatorname{rank}_R H_q(X; R)$ *is defined to be the* q*-th* Betti number β_q *of* X*. The* Euler characteristic $\chi(X)$ *is the alternating sum of Betti numbers,*

$$\chi(X) = \sum_q (-1)^q \beta_q. \tag{1}$$

This is all well and good, except that the concept of rank of an R-module is a bit dubious. Digging into one's favorite algebra textbook sheds some light on this idea.

First, we need the notion of a *free* R-module. We say that an R-module M is *free* over a subset A if for every $x \in M$ there exist unique $r_1, \ldots, r_n \in R$ and unique $a_1, \ldots, a_n \in A$ such that $x = r_1 a_1 + \ldots + r_n a_n$. We say that A is a set of (free) generators, and the rank of M is the cardinality of A. This notion is sufficiently subtle that it is worth consulting a good algebra text for examples, etc.

For an R-module M that isn't free, quotient by the *torsion* submodule (that is, the submodule $\{x \in M | rx = 0 \text{ for some } r \in R\backslash 0\}$). The result is free, and we take the rank of this module to be the rank of M. We'll explore this more when we compute examples.

We note that defining the *relative Euler characteristic* $\chi(X, A)$is simply a matter of replacing homology groups with relative homology groups. Such a gadget is useful because we have the following lemma.

Lemma 1.2. *Where defined, we have*

$$\chi(X) = \chi(A) + \chi(X, A). \tag{2}$$

The proof of the above lemma comes from the long exact sequence of the pair (X, A) and an algebraic lemma which states that for an exact sequence the alternating sum of ranks in that sequence is zero.

The primary utility of relative Euler characteristic is that it allows for the computation of Euler characteristic for adjunction spaces.

Corollary 1.3. *If* Z *is obtained from* Y *by attaching an* n*-cell, and* $\chi(Y)$ *is defined, then we get*

$$\chi(Z) = \chi(Y) + (-1)^n. \tag{3}$$

Proof. The proof is easy enough. We see that $H_q(Z, Y) \cong H_q(E^n, S^{n-1})$ since (E^n, S^{n-1}) is a collared pair. Thus, $\chi(Z, Y) = \chi(E^n, S^{n-1})$. But $\chi(E^n, S^{n-1}) = \chi(E^n) - \chi(S^{n-1}) = 1 - (1 + (-1)^{n-1}) = (-1)^n$, where the latter computation comes because we know the homology of E^n and S^{n-1}. □

So what does this get us? Well, it tells us that the Euler characteristic for a spherical complex is just the alternating sum of the number of 0-cells, 1-cells, etc. Thus, we can immediately read off the Euler characteristic of any spherical complex knowing only which cells compose it.

1.1. **Some examples.**

1.1.1. *Spheres.* As the homology (with $\mathbb{Z}$ coefficients) of S^n has rank one in the 0-th and n-th homology and zero elsewhere,

$$\chi(S^n) = 1 + (-1)^n = \begin{cases} 0, n \text{ odd} \\ 2, n \text{ even} \end{cases} . \tag{4}$$

A consequence is that if N, E, F represent the number of $0, 1$, and 2 cells in any cell complex (we check the next section for a definition) homeomorphic to S^2 (for concreteness, polyhedra in 3-space in which each face meets another in an edge or a vertex are such objects), then we obtain Euler's formula

$$N - E + F = 2. \tag{5}$$

1.1.2. *Projective Space.* With $\mathbb{Z}$ coefficients, we see that projective space has the homology

$$H_q(\mathbb{P}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z}, q = 0, n \text{ (if } n \text{ odd)} \\ 0, q \text{ even and positive} \\ \mathbb{Z}_2, q \text{ odd}, q < n \end{cases} . \tag{6}$$

The $\mathbb{Z}$-rank of the 0-th (and n-th if n odd) homology is 1 (because these modules are free). For the q-th homology modules, q odd (not 0 or n), the module is entirely torsion (note that multiplication by 2 annhilates all of $\mathbb{Z}_2$), so there is no free submodule. Of course, the rank for q even (not 0 or n), is trivially 0. Thus, we get

$$\chi(\mathbb{P}^n; \mathbb{Z}) = \begin{cases} 0, n \text{ odd} \\ 1, n \text{ even} \end{cases} . \tag{7}$$

We could, however, compute over a different ring, say $\mathbb{Z}_2$. Here, we get the homology

$$H_q(\mathbb{P}^n; \mathbb{Z}_2) = \mathbb{Z}_2, \qquad q = 0, \ldots, n. \tag{8}$$

Here, the homology is the same in each dimension (up to n), and indeed is free over one generator. Thus, the rank is one in each dimension. This is a little disconcerting if one is used to computing rank over $\mathbb{Z}$, where we would recognize $\mathbb{Z}_2$ as torsion. But note that since we're working over $\mathbb{Z}_2$ (i.e. our modules are now considered to be $\mathbb{Z}_2$ modules), the torsion submodule are those elements which are annhilated by non-zero members of $\mathbb{Z}_2$. However, the only non-zero member of $\mathbb{Z}_2$ is $\bar{1}$, and it only annhilates 0. In any case, the Euler characteristic is thus $\sum_{i=0}^{n} (-1)^i$, which is again given by (7)

For completeness, we can compute the Euler characteristic over $\mathbb{Q}$. We find the homology is:

$$H_q(\mathbb{P}^n;\mathbb{Q}) = \begin{cases} \mathbb{Q}, q = 0, n \\ 0, \text{otherwise} \end{cases} . \tag{9}$$

The rank here for each homology is clearly 1 in dimension 0 (and n if n odd), and zero otherwise. It is important to note that as a $\mathbb{Q}$-*module* $\mathbb{Q}$ is finitely generated (any nonzero element is a generator) even though as a ring $\mathbb{Q}$ is *not* finitely generated. Note that though both $\mathbb{Q}$ and $\mathbb{Z}_2$ are fields, they have very different behaviors when used to compute homology for $1 \leq q \leq n-1$. Nonetheless, we find that the Euler characteristic for $\mathbb{P}^n$ is 1 if n even, $1 + (-1) = 0$ if n odd, which is the same as found in (7).

Without computing over more exotic PIDs, we might wonder at this point whether the Euler characteristic actually depends on the ring. The answer is in fact that the Euler characteristic is independent. This is clear in the case of a spherical complex, as using any PID the Euler characteristic is just the alternating sum of the number of 0-cells, 1-cells, etc. (this proof only relied on being able to calculate the Euler characteristic of E^n and S^{n-1}, which is easily seen to be independent of the PID).

So, use any PID you want to calculate the Euler characteristic; the result remains the same.

2. Finite Cell Complexes and Mapping Cones

2.1. Finite Cell Complexes and Covering Spaces. We now restrict our attention to finite cell complexes. In what follows, I've lifted the definition straight from G&H. The key point about finite cell complexes is that they are composed of a finite union of closed sets which are each homeomorphic to some unit ball, and further that each of these closed sets meets the others only along common boundaries.

Definition 2.1. *A* finite cell complex *is a spherical complex Z subject to the following restrictions. Z is a compact, Hausdorff space with a finite collection of closed subsets c_j^q (q here denotes dimension, j is an index over some finite index set J_q). We set*

$$Z^q = \cup \left\{ c_j^p | j \in J_p, p \leq q \right\},$$
$$Z^{-1} = \emptyset,$$

and

$$f_j^q = c_j^q \cap Z^{q-1}.$$

We then require that

(1) *if* $c_i^p - f_i^p$ *intersects* $c_j^q - f_j^q$ *then* $p = q$ *and* $i = j$
(2) $Z = \cup_q Z^q$

(3) *for every* c_j^q *there is a map* $\phi_j^q : E^q \to Z$ *sending* S^{q-1} *onto* f_j^q *and mapping* $E^q - S^{q-1}$ *homeomorphically onto* $c_j^q - f_j^q$.

One nice feature of this type of space is that covering spaces are again finite cell complexes.

Theorem 2.2. *Let* X *be a finite cell complex,* $E \xrightarrow{p} X$ *a* d*-fold covering space,* $d > 0$*. Then* E *has a structure of a finite cell complex for which the map* p *is cellular. Moreover*

$$\chi(E) = d\chi(X) \tag{10}$$

(We say p *is* cellular *if* p *maps the* q*-skeleton of* E *into the* q*-skeleton of* X*, for all* $q \geq 0$*.)*

The proof is really just noting that since each point in X has d preimages in E then the cell containing the point lifts to d cells in E.

Proof. For any point $e \in E$ we set $x = p(e)$. As X is a finite cell complex, we can assume that x lies in some unique $c_j^q - f_j^q$. We let $y = (\phi_j^q)^{-1}(x)$. We then note that we can lift ϕ_j^q to E because E^q is simply connected, so we denote by $\psi_{j,1}^q : (E^q, y) \to (E, e)$ the lift of ϕ_j^q. We note that by looking at the other $d-1$ points in the fiber $p^{-1}(x)$ we can similarly generate $\psi_{j,2}^q, \ldots, \psi_{j,d}^q$. Observe that this set of maps does not depend on our choice of x.

If we now let $c_{j,i}^q$ be the image of E^q under $\phi_{j,i}^q$ then we note that we've given a finite cell complex structure to E. Moreover, for each q-cell in X we've generated d q-cells in E, so it is then immediate that $\chi(E) = d \cdot \chi(X)$. □

2.2. **Mapping Cones.** We recall that the mapping cone construction is really an adjunction space construction. Writing $CX = X \times I / X \times \{0\}$, we identify X with the subspace $X \times \{1\}$. The resulting pair (CX, X) is collared, so we can define the *mapping cone* Cf of $f : X \to Y$ to be the adjunction space $CX \cup_f Y$. Let e be the embedding of Y in Cf, i.e. $e : Y \to Cf$.

There are lots of interesting properties of the mapping cone, but we'll list only some.

For adjunction spaces we already have the general long exact sequence (19.15 in Greenberg)

$$\ldots \to H_q(A) \to H_q(Y) \oplus H_q(X) \to H_q(Z) \to H_{q-1}(A) \to \ldots, \tag{11}$$

where X, A, Z are as usual in forming adjunction spaces.

For the mapping cone Cf, we know that the homology of CX is zero for $q > 0$, so we get a long exact sequence

$$\ldots \to H_q(X) \xrightarrow{H_q(f)} H_q(Y) \xrightarrow{H_q(e)} H_q(Cf) \to H_{q-1}(X) \to \ldots. \tag{12}$$

To see that the above maps are as claimed, it is worthwhile at this point to back up and trace through the construction of this sequence. The key fact needed is that (CX, X) is a collared pair, so we know that the map

$$H_q(f) : H_q(CX, X) \to (Cf, Y) \tag{13}$$

is an isomorphism for all q. By the naturality of the long exact sequence of a pair we see that

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_q(X) & \longrightarrow & H_q(CX) & \longrightarrow & H_q(CX,X) & \longrightarrow & H_{q-1}(X) & \longrightarrow \cdots . \\
 & \downarrow{\scriptstyle H_q(f)} & & \downarrow{\scriptstyle H_q(\hat{f})} & & \downarrow{\scriptstyle \cong} & & \downarrow & \\
\cdots \longrightarrow & H_q(Y) & \xrightarrow{H_q(e)} & H_q(Cf) & \longrightarrow & H_q(Cf,Y) & \longrightarrow & H_{q-1}(Y) & \longrightarrow \cdots
\end{array}$$

By our previous remark we note that every third vertical map is an isomorphism, so the Barratt-Whitehead lemma applies. Now, the sequence we obtain via the B-W lemma is (12). The map from $H_q(X) \to H_q(Y) \oplus H_q(CX) \cong H_q(Y)$ is $H_q(f) \oplus 0$ which is just $H_q(f)$ in our identification. The map from $H_q(Y) \oplus H_q(CX)$ is $H_q(e) \oplus -H_q(\hat{f})$ (here, $\hat{f}\colon CX \to Cf$ is the canonical extension of f). But $H_q(\hat{f}) = 0$ since the homology of CX is trivial, so we see that the maps in (12) are as claimed. While this sequence is easily seen to be functorial for strictly commuting pairs

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\
X' & \xrightarrow{f'} & Y'
\end{array} ,$$

it turns out that this sequence is in fact functorial for commutativity up to homotopy, i.e. $f'\alpha \simeq \beta f$. More specifically, we have that

Theorem 2.3. *If* $f'\alpha \simeq \beta f$ *then there exists* $\gamma : Cf \to Cf'$*, an extension of* $e'\beta$*, such that the following commutes:*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_q(X) & \xrightarrow{H_q(f)} & H_q(Y) & \longrightarrow & H_q(Cf) & \longrightarrow & H_{q-1}(X) & \longrightarrow \cdots . \\
 & \downarrow{\scriptstyle H_q(\alpha)} & & \downarrow{\scriptstyle H_q(\beta)} & & \downarrow{\scriptstyle H_q(\gamma)} & & \downarrow{\scriptstyle H_q(\alpha)} & \\
\cdots \longrightarrow & H_q(X') & \xrightarrow{H_q(f')} & H_q(Y') & \longrightarrow & H_q(Cf') & \longrightarrow & H_{q-1}(X') & \longrightarrow \cdots
\end{array}$$

The important thing to note is that the induced map γ may not be unique, and indeed $H_q(\gamma)$ need not be equal to $H_q(\gamma')$ if $\gamma \neq \gamma'$.

We now discuss the homotopy properties of mapping cones.

Theorem 2.4. $f : X \to Y$ *is null-homotopic iff* f *extends to* $F : CX \to Y$.

Proof. First, observe that if f extends then $f = F \circ i$, where $i : X \to CX$ is the inclusion map. But CX is contractible so i is null-homotopic, hence f is. Conversely, if f is null-homotopic then we get a homotopy $F : X \times I \to Y$ with $F(\cdot, 1) = f$ and $F(\cdot, 0) = y_0$ for some $y_0 \in Y$. Applying the quotient map to F yields a valid map from $CX \to Y$ extending f. □

Corollary 2.5. $ef : X \to Cf$ *is null-homotopic.*

Proof. We note the inclusion $i : CX \to CX \cup_f Y$ is an extension of ef, so our result follows from above. □

Mapping cones are most useful homotopically in that they allow us to detect null-homotopy.

Theorem 2.6. $f : X \to Y$ *is null-homotopic iff* Y *is a retract of* Cf.

Proof. By our previous theorem, if f is null-homotopic then it extends to $F : CX \to Y$. Note that since the inclusion $i : Y \to Y$ agrees with F along $X \subset Y$ then we can produced a continuous map from $Cf \to Y$ which is the "gluing" of these two maps (We note that the existence of such a map requires proof, but the proof is fairly straight forward. See page 140 in G&H for a proper statement of the necessary lemma.) As this map takes $Y \subset Cf \to Y$ identically, it provides our retraction.

Conversely, if we have a given retraction $r : Cf \to Y$ then we see that $f = ref$. But we already showed that ef is null-homotopic, so hence f is. □

We can recognize a number of familiar spaces as mapping cones. All the surfaces we've come across thus far S^2, T_g, U_h are mapping cones. The reason is that the construction of attaching a cell is a mapping cone construction, i.e. $C(S^1) \approx E^2$. In fact, any space obtained by attaching a cell can be viewed as a mapping cone.

2.2.1. *Examples.* As a consequence, theorem 2.6 is applicable to a wide collection of spaces. We shall illustrate two applications.

Corollary 2.7. $\mathbb{P}^{n-1}$ *is not a retract of* $\mathbb{P}^n$.

Proof. We observe that $\mathbb{P}^n$ can be identified as a mapping cone, i.e. $\mathbb{P}^n \cong Cf$ where $f : S^{n-1} \to \mathbb{P}^{n-1}$ is the covering space map. But we've shown f is not null-homotopic, so theorem 2.6 applies. □

Corollary 2.8. $S^1 \vee S^1$ *is not a retract of* $S^1 \times S^1$.

Proof. We view the torus as a mapping cone where $f : S^1 \to S^1 \vee S^1$ is the attaching map. We know that $\pi_1(S^1 \vee S^1)$ is free on two generators (from Van Kampen, say), and further we know that f represents the commutator ($aba^{-1}b^{-1}$ with a, b the two generators) in π_1, so specifically f is not null-homotopic. Hence, theorem 2.6 applies. □