ALL ABOUT ORIENTATION

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We restrict our attention to manifolds, so any space mentioned will be an n-dimensional manifold. X is a manifold means every point $x \in X$ has an open neighborhood homeomorphic to $\mathbb{R}^n$ (or, equivalently, B^n, the open unit ball).
In the simple case where our manifold is a surface embeddable in $\mathbb{R}^n$, we can think of an orientation as defining a normal vector at each point through the right hand rule. That is, we choose either a clockwise or counterclockwise loop at each point to provide an orientation for trying to map out a piece of the surface. If we can put all these pieces together in a clear fashion, the surface will be orientable. Unfortunately, the generalization of this to arbitrary rings and dimensions is not visualizable.

The first lemma uses excision to show that

$$H_n(X, X - x; R) \cong R \tag{1}$$

We are generally going to be interested in homology groups isomorphic to R.
A local R-orientation (orientation at a point) is a selection of one of the generators of $H_n(X, X - x; R)$. Note that although this is isomorphic to R, we cannot simply select a generator of R. What matters is the particular element we can pick from this homology, not the overall (known) structure.
We find that we can always extend our R-orientation at a point x to a neighborhood U of x, by finding the right U so that inclusion induces an isomorphism on homology. That is, for any $y \in U$:

$$j_y^U : H_n(X, X - U; R) \to H_n(X, X - x; R) \tag{2}$$

is an isomorphism. We can always find such a neighborhood; in particular, this tells us that we can find a neighborhood U of any point so that

$$H_n(X, X - U; R) \cong R \tag{3}$$

We will generally want to choose such neighborhoods in trying to define a global orientation. The issue in question is whether we can find neighborhoods whose orienations agree on the intersection. We are ready to define a global R-orientation, which has several parts.
1. An open cover U_i
2. Elements $\alpha_i \in H_n(X, X - U_i; R)$ (generally chosen to be generators) so that
i. $j_y^{U_i}(\alpha_i)$ generates $H_n(X, X - y; R)$. (so we actually get a local R-orientation)
ii. For any $x \in U_i \cap U_j$, $j_x^{U_i}(\alpha_i) = j_x^{U_j}(\alpha_j)$ (so local R-orientations match up)

Note that the use of i and j implies nothing about countability. Under these conditions, we have a consistent R-orientation at each point given by $j_y^{U_i}(\alpha_i)$, where $y \in U_i$. In the special case of a $\mathbb{Z}$-orientation, we leave off the $\mathbb{Z}$ and just call it an orientation. This definition is a pain to deal with, so we will almost never use it to decide if some manifold is orientable. The tools we are about to develop will be much more useful in checking orientability.

We are able to break spaces up into connected components as usual: a manifold X is R-orientable iff its connected components are, so we need only look at results for connected manifolds. A useful result on this is

Theorem 22.14: *Any connected non-orientable manifold X has an orientable 2-fold connected cover E.*

The basic idea behind this construction is pretty easy: we let
$E = \{(x, \alpha_x) : x \in X, \alpha_x \text{ generates } H_n(X, X - x)\}$.

Then the obvious projection map $(x, \alpha_x) \mapsto x$ looks like it will provide a 2-fold cover: since $H_n(X, X - x) \cong \mathbb{Z}$, it will have 2 possible generators at each point. The difficult part to see is the topology on E.

We claim that a base for the topology is provided by sets of the form:
$\langle U, \alpha_U \rangle = \{(x, \alpha_x) : x \in U, \alpha_x = j_x^U(\alpha_U)\}$
where $U \subset X$ is open and α_U is a local orientation along U.

We need to check that these sets hit all points and that we can always find a smaller set contained in some neighborhood.

We use the locally constant lemma for both.

Since every neighborhood of any point contains some smaller neighborhood along which we can have a local orientation, our base will hit every (x, α_x).

Now, suppose $\langle U, \alpha_U \rangle \cap \langle U', \alpha_{U'} \rangle$ is a neighborhood of some point (x, α_x).

Then $U \cap U'$ is a neighborhood of x, so we can find some smaller neighborhood V with a local orientation along V.

Thus $\langle V, \alpha_V \rangle \subset \langle U, \alpha_U \rangle \cap \langle U', \alpha_{U'} \rangle$.

In fact, we can always get a 2-fold orientable cover. However, if X were orientable, E would have 2 connected components because we would have separate global orientations $\langle U_i, \alpha_{U_i} \rangle$ and $\langle U_i, -\alpha_{U_i} \rangle$ that would not intersect. This provides us with a method for checking orientability. If we find a connected orientable cover, then our space X is not orientable. On the other hand, if our orientable cover has 2 connected components, then X is orientable.

Corollary *Every connected manifold whose fundamental group has no subgroup of index 2 is R-orientable. In particular, every simply connected manifold is R-orientable.*

Proof. To see this, we have to recall a much earlier result from 5.6: for any path connected cover E with finite fibres, the number of points in the fibre is equal to the index of the subgroup $p_* \Pi_1(E, e_0)$ in $\Pi_1(X, x_0)$. Then any X whose fundamental group has no subgroup of index 2 cannot have a connected 2-fold cover E, which means it must be orientable. □

Example: The Klein bottle K is not orientable, as we will later see easily since $H_2(K) = 0$. Thus we will have a 2-fold connected cover. The Torus will provide such a cover. To see this, we note that the torus is can be viewed as 2 copies of K with one flipped upside down (example 22.16 on page 163). Thus the inverse image of some open set in K will be 2 copies of that set, 1 in each of the 2 copies of K forming the torus.

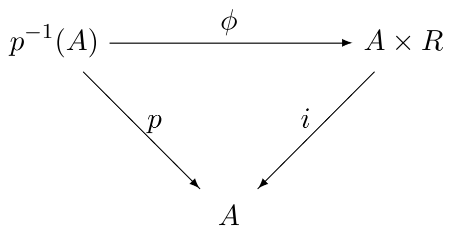
We construct a similar covering space without restricting to generators of the ring. Define $X^0 = \{(x, \alpha_x) : x \in X, \alpha_x \in H_n(X, X - x)\}$ This new covering space is called the R-orientation sheaf of X. We want to construct a map from X^0 to R/units. The book checks the details on this for $R = \mathbb{Z}$, where $v^{-1}(1) = E$ and the only units are ± 1. For more general finitely generated rings, we can do the following.

22.20 *Construct such a map with the property that $v^{-1}(q)$ is open in X^0 and $v^{-1}(q) \to X$ is an m-fold covering space, where $m = |R^\times|$ (number of multiplicative units in R).*

Proof. First we note that the first part is checking that the map v is actually continuous, where R/units is given the discrete topology.
Identify ring elements a and b if $a = ub$, where u is a unit, so that we obtain a set of equivalence class: R modulo units. For any pair $(x, \alpha_x) \in X^0$, α_x will be some multiple of a generator $\beta_i \in H_n(X, X - x; R)$. Say $\alpha_x = r\beta_i$.
Define a map $v : X^0 \to R$ modulo units by $v((x, \alpha_x)) = [r]$.
This will be well defined since it is independent of the generator chosen. Any other generator will be some unit multiple of the first:
$\beta_j = u\beta_i$, so we would have $\alpha_x = ru^{-1}\beta_j$ and $v((x, \alpha_x)) = [ru^{-1}] = [r]$ (equivalence class of r).
As with integers, that $v^{-1}([q])$ is open in X^0 follows directly from lemma 22.4, which lets us find an open neighborhood around any point.
Let's look at $v^{-1}([q]) = \{(x, qu\beta_i) : u \text{ is a unit}\}$ and the projection map $p : X^0 \to X$.
Pick some neighborhood $V \subset X$. $p^{-1}(V) \cap v^{-1}([q]) = \{\langle V, qu_1\alpha_V\rangle, \ldots, \langle V, qu_m\alpha_V\rangle\}$
where $u_1, \ldots, u_m$ are the units of R (finitely many because R is finitely generated) and α_V is some local orientation along V. By the same reasoning as for the covering space E above, we see that p maps each $\langle V, qu_i\alpha_V\rangle$ homeomorphically onto V, providing an m-fold cover. □

Example Say $R = \mathbb{Z}_6$. Then $R^\times = \{\bar{1}, \bar{5}\}$, so R/units has order 3, making it isomorphic to $\mathbb{Z}_3$. Thus we have 2 nonzero elements of units, each of whose inverse image will provide a 2-fold cover of any space X. If this cover is connected, X is not $\mathbb{Z}_6$-orientable.

We now look at sections over some subspace A:
continuous maps $s : A \to X^0$ s.t. $ps =$ inclusion (p is still the projection map $X^0 \to X$), the set of which is denoted ΓA. We say X is R-orientable along A if there is a section that maps A into $v^{-1}(1)$. We are able to get some neat results using sections.
Proposition 22.22 X is R-orientable along A iff there is a homeomorphism $\phi : p^{-1}(A) \to A \times R$, where R is given the discrete topology, s.t. the following diagram commutes.



In this case, $\Gamma A \cong \{$cont maps $A \to R)$, which is in turn isomorphic to R^k when A has k connected components.

Proof. $\Rightarrow$. Suppose we have a section $s : A \to v^{-1}(1)$, writing $s(x) = (x, s'(x))$.
In other words, for each $x \in A$, $s'(x)$ actually generated $H_n(X, X - x; R)$.
Given any pair $(x, \alpha_x) \in p^{-1}(A)$, $\exists! \lambda_x \in R$ s.t. $\alpha_x = \lambda_x s'(x)$ (this is just saying that any module element is a unique multiple of a particular generator).
Now we can define $\phi(x, \alpha_x) = (x, \lambda_x)$. Identity on the first coordinate is clearly a continous, invertible bijection. Our mapping between α_x and λ_x is an isomorphism since
$H_n(X, X - x; R) \cong R$. Since both sets are given the discrete topology, it will provide a homeomorphism. The locally constant lemma (22.4) allows us to be sure that $\lambda_y = \lambda_x$ for all y within some neighborhood U of x (by having j_y^U an isomorphism for $y \in U$), so that a homeomorphism on each component gives an overall homeomorphism. Thus the overall map ϕ is a homeomorphism. Once we have this homeomorphism and see that the x coordinate is preserved, commutativity becomes obvious.

$\Leftarrow$. Suppose we have a homeomorphism $\phi : p^{-1}(A) \to A \times R$. Define $s(x) = \phi^{-1}(x, 1)$. We check that this is a section: continuity is automatic from ϕ. $ps(x) = p(\phi^{-1}(x, 1)) = \pi(x, 1) = x$ by commutativity of the digram, so we see that $ps =$ inclusion as needed. $\square$

In particular, for this result and below, we can choose A to be all of X and get results about R-orientability. Note that $X \times R$ is not connected, so we again see that for orientable manifolds we get a disconnected cover X^0. This is a big deal. Sections are going to be one of our most useful tools for orientability. Let's see what we can do next.

It is not obvious that the map we get from sections, $j_A : H_n(X, X - A; R) \to \Gamma A$, is continuous; nor is it obvious that the range of j_A for any closed $A \subset X$ will be only the sections on A with compact support. However, these facts come together in theorem 22.24, which is good to have seen but not worth reproving. We get the following very useful results:
i. $H_q(X, X - A) = 0$ for $q > n$: we have no homology past the dimension of the manifold.
22.25: If A is connected and non-compact, $H_n(X, X - A; R) = 0$.
22.28: if X is compact connected and the only unit in R that fixes nonzero elements is the identity, we get $H_n(X) \cong \begin{cases} R & X \text{ R-orientable} \\ 0 & \text{else} \end{cases}$

Problem *Prove that if X is $\mathbb{Z}_3$ orientable, then X is $\mathbb{Z}$ orientable.*

Proof. Assume X is $\mathbb{Z}_3$ orientable: there exists a global $\mathbb{Z}_3$ orientation, $\{(U_i, \alpha_i)\}$, where α_i is a generator of $H_n(X, X - U_i; \mathbb{Z}_3)$, such that if $x \in U_i \cap U_j$ then $j_x^{U_i}(\alpha_i) = j_x^{U_j}(\alpha_j)$.
Suppose for contradiction that X is not $\mathbb{Z}$ orientable. Then for any set of α_i', generators of $H_n(X, X - U_i; \mathbb{Z})$ we can find some i, j, and x such that $x \in U_j \cap U_j$, and $j_x^{U_i}(\alpha_i') \neq j_x^{U_j}(\alpha_j')$.
Note that from a previous problem, we have a map
$\theta : H(X, X - U; \mathbb{Z}) \to H(X, X - U; \mathbb{Z}_3)$
that takes $\{\sum a_i\sigma_i\} \in H(X, X - U; \mathbb{Z})$ to $\{\sum \pi(a_i)\sigma_i\} \in H(X, X - U; \mathbb{Z}_3)$
In particular, the generators are in 1-1 correspondence under the map θ: a generator $\{\sigma\} \in H(X, X - U; \mathbb{Z})$ maps to $\{\sigma\} \in H(X, X - U; \mathbb{Z}_3)$ and $\{-\sigma\}$ maps to $\{2\sigma\}$ since $\pi(-1) = \overline{2}$. Thus we can choose α_i' such that $\theta(\alpha_i') = \alpha_i$. Let i, j, x be chosen to demonstrate non-orientability as above. Then we have

$$\begin{array}{ccc} H(X, X - U_i; \mathbb{Z}) & \xrightarrow{\theta} & H(X, X - U_i; \mathbb{Z}_3) \\ \downarrow j_x^{U_i} & & \downarrow j_x^{U_i} \\ H(X, X - x; \mathbb{Z}) & \xrightarrow{\theta} & H(X, X - x; \mathbb{Z}_3) \\ \uparrow j_x^{U_j} & & \uparrow j_x^{U_j} \\ H(X, X - U_j; \mathbb{Z}) & \xrightarrow{\theta} & H(X, X - U_j; \mathbb{Z}_3) \end{array}$$

Because we know both $j_x^{U_i}$ and θ map generators to generators (θ in the particular way above), we have that $j_x^{U_i}(\alpha_i') \neq j_x^{U_j}(\alpha_j') \Rightarrow \theta(j_x^{U_i}(\alpha_i')) \neq \theta(j_x^{U_j}(\alpha_j'))$.
But $\theta(j_x^{U_i}(\alpha_i')) = j_x^{U_i}(\theta(\alpha_i')) = j_x^{U_i}(\alpha_i) = j_x^{U_j}(\alpha_j) = \theta(j_x^{U_j}(\alpha_j'))$, a contradiction. Thus X must be $\mathbb{Z}$-orientable. $\square$

We could extend this to say $\mathbb{Z}_n$-orientability for $n > 2$ implies orientability. Though we don't have all the machinery yet, a more general result will be that
X is orientable iff it is R-orientable for all rings R with at least 2 units.

Problem Show that any manifold X that is a topological group is orientable.

Proof. This problem was on the midterm, so you should know the basic ideas already. Constructing a detailed proof is rather long and painful, but here it is. Much of the work is in the following lemma (homework problem on its own)
Lemma Let U be a path connected neighborhood of the identity in X. Define $\theta_g : X \to X$ as multiplication by g. Then for $x \in U \cap gU$, we get a homotopy commuting diagram:

$$\begin{array}{ccc} (X, X-U) & \xrightarrow{\quad i \quad} & (X, X-x) \\ & \searrow^{\theta_g} \quad {}^{j}\nearrow & \\ & (X, X-gU) & \end{array}$$

i.e. $j\theta_g \simeq i$ (homotopy of pairs), where i and j are inclusions.

Proof. We first note that θ_g is a homeomorphism (1-1 onto, as is $\theta_g^{-1} = \theta_{g^{-1}}$, multiplication on the left by g^{-1}). There are several things to show, each of which is made more awkward by the fact that we have no assumption about the group being abelian. Basically, I had to play around with things to figure out when I needed right versus left cosets and such.
Pick $y \in X - U$. We are going to need to select a particular path from y to gy, the choice of which will only make sense after checking that it works.
Since U is path connected, $U^{-1} = \{u^{-1} : u \in U\}$ will also be path connected, as will $xU^{-1}y$.
We want to let $\sigma(t) \in xU^{-1}y$ be a path from y to gy (this is the magic choice).
We need to verify that both y and gy are in $xU^{-1}y$.
$x \in U \;\Rightarrow\; x^{-1} \in U^{-1} \;\Rightarrow\; x(x^{-1})y = y \in xU^{-1}y$.
$x \in gU \;\Rightarrow\; g^{-1}x \in U \;\Rightarrow\; x^{-1}g \in U^{-1} \;\Rightarrow\; x(x^{-1}g)y = gy \in xU^{-1}y$.
Since this path exists, we use it to define our desired homotopy.
For any $z \in X$, we can write $z = y(y^{-1}z)$
Define $F : X \times I \to X$ by $F(z,t) = \sigma(t)y^{-1}z$.
Check: $F(z,0) = \sigma(0)y^{-1}z = y(y^{-1}z) = z$.
$F(z,1) = \sigma(1)y^{-1}z = gy(y^{-1}z) = gz$.
Since it involves a continuous path, F is continuous with respect to t. Since multiplication by any group element is continous, F is continuous with respect to z. Thus F is a homotopy.
We still have to check that it's a homotopy of pairs: $z \in X - U \;\Rightarrow\; F(z,t) \in X - x$.
It is easier to check the contrapositive of this: $F(z,t) = x \;\Rightarrow\; z \in U$.
Assume $\sigma(t)y^{-1}z = x$ for some t. Since we chose $\sigma(t)$ to be in $xU^{-1}y$, this means
$x \in (xU^{-1}y)y^{-1}z = xU^{-1}z \;\Rightarrow\; 1 \in U^{-1}z \;\Rightarrow\; z \in U$ as desired. Thus we have verified the lemma. □

We are now ready to construct a global orientation for X. We pick a path connected neighborhood V of the identity homeomorphic to B^n such that j_x^U is an isomorphism for all $x \in U$ (possible by lemma 22.4: locally constant lemma, summarized above).
We choose our open cover to be $\{gU : g \in X\}$. After selecting α_U as some generator of $H_n(X, X-U)$, we define $\alpha_g = H_n(\theta_g)\alpha_U \in H_n(X, X-gU)$.
Since θ_g is a homeomorphism for any g, it always induces an isomorphism on homology, so α_g will generate $H_n(X, X-gU)$.
We're ready to pull in our lemma for the first time, which after applying homology tells us:
$H_n(i) = H_n(j\theta_g) = H_n(i)H_n(\theta_g)$.
We can rewrite this with the usual notation for inclusions: $j_x^U = j_x^{gU} H_n(\theta_g)$.

Now it becomes obvious that, since j_x^U and $H_n(\theta_g)$ are isomorphisms, j_x^{gU} must also be, so we at least have well-defined local orientations.
The only detail left to check is that our local orientations agree on the intersection.
Suppose $y \in gU \cap hU$. Then $j_y^{gU}(\alpha_g) = j_y^{gU}(H_n(\theta_g)\alpha_U) = (j_y^{gU} H_n(\theta_g))(\alpha_U) = j_y^U(\alpha_U) = (j_y^{hU} H_n(\theta_h))(\alpha_U) = j_y^{hU}(H_n(\theta_h)\alpha_U) = j_y^{hU}(\alpha_h)$ by our aforementioned result from applying homology to the lemma (to both θ_g and θ_h, since g was arbitrary in the lemma). $\square$

These are the major useful results that orientability gives us. We get information about the top level homology from it, which can help us distinguish between spaces. Unfortunately, orientability is not invariant under homotopy equivalence, because the dimension of the manifold can be changed (even if the space remains a manifold). For instance, the open mobius band is homotopically equivalent to the circle, but only the latter is orientable (by 22.28). Greenberg and Harper provide a table of which familiar spaces are orientable on page 168. We want spaces to be orientable, because that allows us to use the power of Poincaré duality. Let's use our various methods of computation to examine a non-orientable manifold.

Example: *for even n, $\mathbb{P}^n$ is not orientable.*
In light of all we have, the easiest way to see this is that $H_n(\mathbb{P}^n) = 0$ for even n. Then since $\mathbb{P}^n$ is compact connected, non-orientability follows from 22.28 (above).
It is still useful to note how this relates to other results. For instance, $\Pi_q(\mathbb{P}^n, x_0) = \mathbb{Z}_2$, which does have a subgroup of order 2 (namely itself). Thus we can't use the corollary to thm 22.14 to say it's orientable.
Going along with this, S^n provides a 2-fold connected cover for $\mathbb{P}^n$, which we know must exist since it's not orientable.
Finally, we can be sure that, for even n, $\mathbb{P}^n$ is not a topological group.