Notes on Cohomology

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1 Singular Cochains

Recall that the singular chains $S_*(X;R)$ form an R-module, where an arbitrary chain $s \in S_q(X;R)$ is a R-linear combination of q simplices, $s = \sum r_i\sigma_i$, $r_i \in R$, with R-module structure given by $rs = r\sum r_i\sigma_i = \sum rr_i\sigma_i$. We define the **singular cochain** on X to be the R module which is the Hom dual of S_*, that is

$$S^q(X;R) = \mathrm{Hom}_R(S_q(X;R), R)$$

the set of R-linear homomorphisms from $S_q(X;R)$ into R. Note that an arbitary $f \in S^q(X;R)$ is of the form

$$\begin{aligned} f : S_q(X;R) &\rightarrow R \\ \sum r_i\sigma_i &\mapsto f(\sum r_i\sigma_i) = \sum r_i f(\sigma_i) \end{aligned}$$

where the R-linearity implies that f is defined only by its action on q-simplices σ_i. In homology we had the boundary maps

$$\rightarrow S_{q+1}(X;R) \xrightarrow{\partial_{q+1}} S_q(X;R) \xrightarrow{\partial_q} S_{q-1}(X;R)$$

We define the **coboundary** map

$$\begin{aligned} \delta^q : S^q(X;R) &\rightarrow S^{q+1}(X;R) \\ f &\mapsto \delta^q f \end{aligned}$$

where $\delta^q f$ is defined on the simplices so that

$$\delta^q f(\sigma) = f(\partial_{q+1}(\sigma))$$

so

$$\delta^q f(\sum r_i\sigma_i) = \sum r_i\delta^q f(\sigma_i) = \sum r_i f(\partial_{q+1}(\sigma_i)$$

Note that the coboundary maps *increase* the degree, where the boundary maps decrease the degree. We then get the sequence

$$\leftarrow S^{n+1}(X;R) \xleftarrow{\delta^n} S^n(X;R) \xleftarrow{\delta^{n-1}} S^{n-1}(X;R) \leftarrow$$

Note that $\delta\delta = 0$ as

$$\delta^{n+1}\delta^n f(\sigma) = \delta^n f(\partial_{n+2}\sigma) = f(\partial_{n+1}\partial_{n+2}\sigma) = f(0) = 0$$

We define the nth cohomology group of X with coefficients in R to be

$$H^n(X;R) = \frac{\ker \delta^n}{\text{im } \delta^{n-1}} = \frac{Z^n(X;R)}{B^n(X;R)}$$

Note that by applying the same idea to the relative chains

$$\cdots \to S_n(X,A;R) \to S_{n-1}(X,A;R) \to \ldots$$

we can get relative cohomology. In fact, the major points of homology: the long exact sequence of a pair, excision, the Mayer-Vietoris Sequence, and the homology of a point, all carry over into cohomology, albeit with arrows reversed.

How does cohomology behave with respect to continuous maps $h : X \to Y$. We have the induced map on chains $S_*(h) : S_*(X;R) \to S_*(Y;R)$ so that

$$\begin{array}{ccc} S_n(X;R) & \xrightarrow{S_n(h)} & S_n(Y;R) \\ \partial_n \downarrow & & \downarrow \partial_n \\ S_{n-1}(X;R) & \xrightarrow{S_{n-1}(h)} & S_{n-1}(Y;R) \end{array}$$

commutes. On the cochain side, we get

$$\begin{array}{ccc} S^n(X;R) & \xleftarrow{S^n(h)} & S^n(Y;R) \\ \delta^{n-1} \uparrow & & \uparrow \delta^{n-1} \\ S^{n-1}(X;R) & \xleftarrow{S^{n-1}(h)} & S^n(Y;R) \end{array}$$

Where we define

$$S^n(h)(f)(\sigma) = f(S_n(h)\sigma)$$

and note

$$\begin{aligned} (\delta^{n-1}S^{n-1}(h)(f))(\sigma) &= (S^{n-1}(h)(f))(\partial_n(\sigma)) \\ &= f(S_{n-1}(h)(\partial_n(\sigma)) \\ &= f(\partial_n S_n(h)(\sigma)) \\ &= \delta^{n-1} f(S_n(h)(\sigma)) \\ &= S^n(h)\delta^{n-1}(f)(\sigma) \end{aligned}$$

so the diagram commutes. Thus, the induced map on cohomolgy $H^n(f)$ reverses the direction of the arrows. (So if $f : X \to Y$, then $H^n(f) : H^n(Y) \to H^n(X)$.)

2 The Ext Functor

A basic tool of homological algebra is the exact sequence. So, a fundamental question is, if we are given a short exact sequence

$$0 \to A \to B \to C \to 0$$

and we apply the Hom functor, is our new seqence

$$0 \leftarrow \text{Hom}(A,R) \leftarrow \text{Hom}(B,R) \leftarrow \text{Hom}(C,R) \leftarrow 0$$

exact? The answer is not necessarily. As an example consider the sequence

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_2 \to 0$$

Then the Hom dual, with $R = \mathbb{Z}_2$

$$\begin{array}{ccccccccc} 0 & \leftarrow & \text{Hom}(\mathbb{Z},\mathbb{Z}_2) & \leftarrow & \text{Hom}(\mathbb{Z},\mathbb{Z}_2) & \leftarrow & \text{Hom}(\mathbb{Z}_2,\mathbb{Z}_2) & \leftarrow & 0 \\ & & \cong \mathbb{Z}_2 & & \cong \mathbb{Z}_2 & & \cong \mathbb{Z}_2 & & \end{array}$$

is not exact. It turns out however that the first two joints are exact, so we have an exact sequence

$$(?) \leftarrow \text{Hom}(A,R) \leftarrow \text{Hom}(B,R) \leftarrow \text{Hom}(C,R) \leftarrow 0$$

So what is $(?)$ that makes this sequence exact? The answer is Ext(C,R). It is somewhat important to note one particular case when Ext(C,R) is zero. If

$$0 \to A \to B \to C \to 0$$

is a *split* short exact sequence then $B = A \oplus C$ and

$$0 \leftarrow \text{Hom}(A,R) \leftarrow \text{Hom}(B,R) \leftarrow \text{Hom}(C,R) \leftarrow 0$$

is exact, as $\text{Hom}(B,R) \cong \text{Hom}(A \oplus C, R) \cong \text{Hom}(A,R) \oplus \text{Hom}(C,R)$. Note that as

$$0 \to S_*(A) \to S_*(X) \to S_*(X)/S_*(A) \to 0$$

splits, we have that

$$0 \leftarrow S^*(A) \leftarrow S^*(X) \leftarrow S^*(X)/S^*(A) \leftarrow 0$$

is a short exact sequence of cochain complexes that induces a long exact sequence on cohomology.

$$\leftarrow H^{q+1}(X) \leftarrow H^{q+1}(X,A) \leftarrow H^q(A) \leftarrow H^q(X) \leftarrow H^q(X,A) \leftarrow$$

To explore the Ext functor in more detail, we require some information about resolutions.

2.1 R-module Resolutions

An **R-module resolution** of an R-module M is a chain complex $\{C_q, \partial_q\}$ and an epimorphism $\varepsilon : C_0 \to M$ such that

$$\ker \partial_q = \text{im } \partial_{q+1}, \ker \varepsilon = \text{im } \partial_1$$

As example, consider the group $\mathbb{Z} \oplus \mathbb{Z}_3$. A resolution of this group could be as follows,

$$0 \to \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z} \oplus \mathbb{Z}_3$$

where ∂_1 is defined by $x \mapsto (0, 3x)$, and ε is defined by $(x,y) \mapsto (x, \pi(y))$ (where π is projection modulo 3). Note that resolutions are not unique, and in fact different resolutions may involve different numbers of maps. The length of a resolution is the minimal length of of a resolution. Note that given a finitely generated abelian group, the fundamental theorem of abelian groups states that it can be written as the direct sum of free parts, and cyclic groups of prime power orders. Using the technique in the above example we can find a length two resolution for any finitely generated abelian group. (We assume, for the remainder of this document, that all abelian groups are finitely generated.)

While it is true that resolutions are not unique, we have the following basic theorem.

Theorem 1. *Given $f : M \to M'$ a homomorphism of R-modules M and M' with resolutions C_* and C'_* respecitvely, then there exists a chain map $f : C_* \to C'_*$ such that the ladder*

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ C_1 & \xrightarrow{f_1} & C'_1 \\ \downarrow & & \downarrow \\ C_0 & \xrightarrow{f_0} & C'_0 \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & M' \end{array}$$

commutes, and any two such chain maps are chain homotopic.

Proof. We construct f_0 by induction. Let $\alpha \in C_0$. Then consider $f\varepsilon\alpha \in M'$. Note that ε' is surjective, so there exists an $\bar{\alpha} \in C'_0$ such that $\varepsilon'\bar{\alpha} = f\varepsilon\alpha$. We merely set $f_0(\alpha) = \bar{\alpha}$, and continue to construct f_i in the same manner.

Let f_0 and g_0 be two such chain maps. We build a chain homotopy between them as follows. We define $D_0 : C_0 \to C'_1$ such that $\partial'_1 D_0 = f_0 - g_0$. Let $\alpha \in C_0$. Then note that $\epsilon'(f_0(\alpha) - g_0(\alpha)) = 0$, as $f(\epsilon(\alpha)) - g(\epsilon(\alpha)) = 0$ as $f = g$ (both f' and g' are extending the same map. This implies that there is an $\bar{\alpha} \in C'_1$ such that $\partial_1(\bar{\alpha}) = f_0(\alpha) - g_0(\alpha)$ and we define $D_0(\alpha) = \bar{\alpha}$. We inductively build D_q such that $\partial'_{q+1} D_q + D_{q-1}\partial_q = f_q - g_q$, so

$$\partial_{q+1} D_q = f_q - g_q - D_{q-1}\partial_q$$

To show that this is a chain homotopy we need to show that (given $c_q \in C_q$)

$$f_q(c_q) - g_q(c_q) - D_{q-1}\partial_q(c_q)$$

is a cycle. Note that

$$\begin{aligned} \partial'_q[f_q(c_q) - g_q(c_q) - D_{q-1}\partial_q(c_q)] &= f_{q-1}\partial_q(c_q) - g_{q-1}\partial_q(c_q) - \partial'_q D_{q-1}\partial_q(c_q) \\ &= f_{q-1}\partial_q(c_q) - g_{q-1}\partial_q(c_q) - (f_{q-1}\partial_q(c_q) - g_{q-1}\partial q(c_q) \\ &\quad - D_{q-2}\partial_{q-1}\partial_q(c_q)) \\ &= 0 \end{aligned}$$

(using our inductive definition of D_{q-1}) and hence we have a cycle and thus D_q is a chain homotopy. □

What is the payoff from this theorem? Well, given R-module M, and N and a resolution $C_* \to M$, we can consider the cochain complex $\mathrm{Hom}_R(C_*, N)$

$$\begin{array}{c} \mathrm{Hom}(C_{q+1}, N) \\ \uparrow \delta^q \\ \mathrm{Hom}(C_q, N) \\ \uparrow \delta^{q-1} \\ \mathrm{Hom}(C_{q-1}, N) \\ \uparrow \end{array}$$

We have that

$$\mathrm{Ext}^q_R(M, N) = \frac{\ker(\delta^q)}{\mathrm{im}\ \delta^{q-1}}$$

Our theorem tells us that this is well define – it does not depend on the choice of resolution of M!

2.2 Computing Ext

We now concern ourselves with investigating computing $\text{Ext}^1_{\mathbb{Z}}(G_1, G_2)$ for finitely generated abelian groups G_1, G_2. Note that $\text{Ext}^1_{\mathbb{Z}}$ in in some sense the 'important' version of Ext that comes up in the univeral coefficient theorem when dealing with abelian groups (as abelian groups are $\mathbb{Z}$ modules) and, given finitely generated abelian groups G_1, G_2 we call

$$\text{Ext}^1_{\mathbb{Z}}(G_1, G_2) = \text{Ext}(G_1, G_2) = \frac{\ker \delta^1}{\text{im } \delta^0}$$

Note that (as noted in our example in the beginning of last section) when we have finitely generated abelian groups we have a resolution of length 2 so that $\text{Ext}^q_{\mathbb{Z}}(G_1, G_2) = 0$, so in fact $\text{Ext}(G_1, G_2)$ is really the only Ext group to concern ourselfs with when dealing with finitely generated abelian groups. The following are useful statements regarding computing Ext.

Proposition 1. *$Ext^n_{\mathbb{Z}}(G_1, G_2) \cong 0$ for $n \geq 2$ if G_1 is a finitely generated abelian group.*

Proof. Note that we have a resolution of length two for G_1, and for that resolution we have that $\delta^i = 0$, for $i \geq 1$. So in particular for $i \geq 2$, $\ker \delta^i = \text{im } \delta^{i-1} = 0$, and hence $\text{Ext}^n_{\mathbb{Z}}(G_1, G_2) \cong 0$ as desired. □

Proposition 2. *$Ext(\mathbb{Z}, G) \cong 0$*

Proof. Note that $0 \to \mathbb{Z} \to \mathbb{Z} \to 0$ is a free resolution of G, so applying Hom we have

$$0 \to \text{Hom}(\mathbb{Z}, G) \xrightarrow{\epsilon^*} \text{Hom}(\mathbb{Z}, G) \xrightarrow{\delta^0} 0$$

and hence im $\delta^0 = \ker \delta^1 = 0$, so $\text{Ext}(\mathbb{Z}, G) \cong 0$ as desired. □

Proposition 3. *$Ext(\mathbb{Z}_n, G) \cong G/nG$*

Proof. Note that we have the free resolution $\mathbb{Z} \xrightarrow{\partial_1 = \cdot n} \mathbb{Z} \to \mathbb{Z}_n$, and applying Hom we have

$$0 \to \text{Hom}(\mathbb{Z}_n, G) \xrightarrow{\epsilon^*} \text{Hom}(\mathbb{Z}, G) \xrightarrow{\delta^0} \text{Hom}(\mathbb{Z}, G) \xrightarrow{\delta^1} 0$$

Note that $\text{Hom}(\mathbb{Z}, G) \cong G$, so $\ker \delta^1 \cong G$. Let $\phi \in \text{Hom}(\mathbb{Z}, G)$, with g such that $1 \mapsto g$. Then $\delta^0\phi$ is defined by

$$\delta^0\phi(1) = \phi(\partial_1(1)) = \phi(n(1)) = n\phi(1) = ng$$

So im $\delta^0 \cong nG$, and hence $\text{Ext}(\mathbb{Z}_n, G) \cong G/nG$ □

Corollary 1. *$Ext(\mathbb{Z}_n, \mathbb{Z}) \cong \mathbb{Z}_n$*

Corollary 2. *$Ext(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathbb{Z}_d$ where $d = gcd(m, n)$*

Proof. We note that by our above theorem we have that $\text{Ext}(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathbb{Z}_m/n\mathbb{Z}_m$. Let $\varphi : \mathbb{Z}_m \to n\mathbb{Z}_m$ defined by $x \mapsto nx$. Then $n\mathbb{Z}_m \cong \mathbb{Z}_m/(\ker \varphi)$. Note that $\ker \varphi = \{x \in \mathbb{Z}_m : nx = mk\}$ and $nx = mk$ iff $\frac{m}{d}$ divides x, so $\ker \varphi = \{0, \frac{m}{d}, 2\frac{m}{d}, \ldots, (d-1)\frac{m}{d}\} \cong \mathbb{Z}_d$, so $n\mathbb{Z}_m \cong \mathbb{Z}_m/\mathbb{Z}_d$. Then

$$\text{Ext}(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathbb{Z}_m/n\mathbb{Z}_m \cong (\mathbb{Z}_m/(\mathbb{Z}_m/\mathbb{Z}_d)) \cong \mathbb{Z}_d$$

as desired. □

These results, along with the following results allows us to compute $\text{Ext}(G, N)$ for any finitely generated abelian group G.

Proposition 4. *$Ext(\bigoplus G_i, N) \cong \bigoplus Ext(G_i, N)$*

Proposition 5. *$Ext(G, \bigoplus N_i) \cong \bigoplus Ext(G, N_i)$*

Sketch of Proof. These follow directly from the fact that if $C_*^{(i)} \to G_i$ are free resolutions of G_i, then $\bigoplus C_*^{(i)} \to \bigoplus G_i$ is a free resolution (where the connecting maps ∂_j are $\bigoplus_i \partial_j^{(i)}$), and from the fact that $\text{Hom}(G, \bigoplus N_i) \cong \bigoplus \text{Hom}(G, N_i)$. □

By the fundamental theorem of abelian groups we can break any abelian group up into the direct sum of torsion parts, and free parts. By the first proposition we have

$$\text{Ext}(\text{free}(G), N) \cong 0$$

Thus

$$\text{Ext}(G, N) \cong \text{Ext}(\text{free}(G), N) \oplus \text{Ext}(\text{torsion}(G), N) \cong \text{Ext}(\text{torsion}(G), N)$$

If G is a finitely generated abelian group, then $\text{torsion}(G) \cong \bigoplus_{i=1}^n \mathbb{Z}_{n_i}$. Thus

$$\text{Ext}(G, N) \cong \text{Ext}(\bigoplus_{i=1}^n \mathbb{Z}_{n_i}, N) \cong \bigoplus_{i=1}^n \text{Ext}(\mathbb{Z}_{n_i}, N) \cong \bigoplus_{i=1}^n N/n_i N$$

In particular we have that

Proposition 6. *$Ext(G, \mathbb{Z}) \cong \bigoplus_{i=1}^n \mathbb{Z}_{n_i} \cong torsion(G)$*

Let's compute some examples.

Examples:

$$\begin{aligned} \text{Ext}(\mathbb{Z} \oplus \mathbb{Z}_{12}, \mathbb{Z}_{20} \oplus \mathbb{Z}) &\cong \text{Ext}(\mathbb{Z}, \mathbb{Z}_{20} \oplus \mathbb{Z}) \oplus \text{Ext}(\mathbb{Z}_{12}, \mathbb{Z}_{20}) \oplus \text{Ext}(\mathbb{Z}_{12}, \mathbb{Z}) \\ &\cong \mathbb{Z}_4 \oplus \mathbb{Z}_{12} \\ \text{Ext}(\mathbb{Z}_{10} \oplus \mathbb{Z}_7, \mathbb{Z}_4 \oplus \mathbb{Z}_{49}) &\cong \text{Ext}(\mathbb{Z}_{10}, \mathbb{Z}_4) \oplus \text{Ext}(\mathbb{Z}_{10}, \mathbb{Z}_{49}) \oplus \text{Ext}(\mathbb{Z}_7, \mathbb{Z}_4) \oplus \text{Ext}(\mathbb{Z}_7, \mathbb{Z}_{49}) \\ &\cong \mathbb{Z}_2 \oplus \mathbb{Z}_7 \end{aligned}$$

3 The Universal Coefficient Theorem

We can compute cohomology groups based on homology groups using the Universal Coefficient theorem.

Theorem 2 (Universal Coefficient Theorem). *There exists a natural, split short exact sequence*

$$0 \to Ext(H_{n-1}(X, A; \mathbb{Z}), G) \to H^n(X, A; G) \to Hom(H_n(X, A; \mathbb{Z}), G) \to 0$$

though the splitting is not necessarily natural.

In particular

$$H^n(X, A; G) \cong \text{Ext}(H_{n-1}(X, A; \mathbb{Z}), G) \oplus \text{Hom}(H_n(X, A; \mathbb{Z}), G)$$

and setting $A = \emptyset$ we have

$$H^n(X; G) \cong \text{Ext}(H_{n-1}(X; \mathbb{Z}), G) \oplus \text{Hom}(H_n(X; \mathbb{Z}), G)$$

Combining this with our previous notes on computing Ext and Hom we get the following.

Corollary 3. $H^n(X;\mathbb{Z}) \cong \textit{torsion}(H_{n-1}(X;\mathbb{Z})) \oplus \textit{free}(H_n(X;\mathbb{Z}))$

Redoing the proof of the universal coefficient, looking instead at modules over a PID R, and replacing Hom with Hom_R we get the natural split short exact sequence.

$$0 \to \text{Ext}_R(H_{n-1}(X;R), G) \to H^n(X;G) \to \text{Hom}_R(H_n(X;R), G) \to 0$$

Note that R being a PID is vital, as a submodules of a free R-module are free if R is a PID, and this was the key fact about abelian groups that allowed us to prove the universal coefficient theorem. In particular if we take R to be a field F, we have that the Ext_F terms are trivial. Thus we have the following

Proposition 7. *If F is a field $H^n(X;F) \cong \text{Hom}_F(H_n(X;F), F)$ and therefire $H^n(X;F)$ and $H_n(X;F)$ are dual vector spaces.*

Note that if $H_q(X;\mathbb{Z})$ is the highest level of nonzero homology, and if $H_q(X;\mathbb{Z})$ is free, then $H^{q+1}(X;G) \cong \text{Ext}(H_q(X;\mathbb{Z}), G) \cong 0$. In particular, if M is an orientable n-manifold, then $H^{n+1}(M;G)$ is zero. If M is non-orientable, however, we cannot neccessarily say that $H^n(M;G) \cong 0$ (as seen in one of our examples below).

We can compute some examples:

Example: The Torus $T = \mathbb{S}^1 \times \mathbb{S}^1$

$$\begin{aligned}
H^q(T;G) &\cong 0 \text{ (for } q \geq 3) \\
H^2(T;G) &\cong \text{Ext}(H_1(T;\mathbb{Z}), G) \oplus \text{Hom}(H_2(T;\mathbb{Z}), G) \cong \text{Ext}(\mathbb{Z} \oplus \mathbb{Z}, G) \oplus \text{Hom}(\mathbb{Z}, G) \\
&\cong G \\
H^1(T;G) &\cong \text{Ext}(H_0(T;\mathbb{Z}), G) \oplus \text{Hom}(H_1(T;\mathbb{Z}), G) \cong \text{Ext}(\mathbb{Z}, G) \oplus \text{Hom}(\mathbb{Z} \oplus \mathbb{Z}, G) \\
&\cong G \oplus G \\
H^0(T;G) &\cong \text{Hom}(H_0(T;\mathbb{Z}), G) \cong \text{Hom}(\mathbb{Z}, G) \\
&\cong G
\end{aligned}$$

Example: $P = \mathbb{R}P^3$ with integer coefficients

$$\begin{aligned}
H^q(P;\mathbb{Z}) &\cong 0 \text{ (for } q \geq 4) \\
H^3(P;\mathbb{Z}) &\cong \text{Ext}(H_2(P;\mathbb{Z}), \mathbb{Z}) \oplus \text{Hom}(H_3(P;\mathbb{Z}), \mathbb{Z}) \cong \text{Hom}(\mathbb{Z}, \mathbb{Z}) \\
&\cong \mathbb{Z} \\
H^2(P;\mathbb{Z}) &\cong \text{Ext}(H_1(P;\mathbb{Z}), \mathbb{Z}) \oplus \text{Hom}(H_2(P;\mathbb{Z}), \mathbb{Z}) \cong \text{Ext}(\mathbb{Z}_2, \mathbb{Z}) \\
&\cong \mathbb{Z}_2 \\
H^1(P;\mathbb{Z}) &\cong \text{Ext}(H_0(P;\mathbb{Z}), \mathbb{Z}) \oplus \text{Hom}(H_1(P;\mathbb{Z}), \mathbb{Z}) \cong \text{Ext}(\mathbb{Z}, \mathbb{Z}) \oplus \text{Hom}(\mathbb{Z}_2, \mathbb{Z}) \\
&\cong 0 \\
H^0(P;\mathbb{Z}) &\cong \text{Hom}(H_0(P;\mathbb{Z}), \mathbb{Z}) \cong \text{Hom}(\mathbb{Z}, \mathbb{Z}) \\
&\cong \mathbb{Z}
\end{aligned}$$

Example: $P = \mathbb{R}P^2$ with integer coefficients

$$\begin{aligned}
H^q(P,\mathbb{Z}) &\cong 0 \text{ (for } q \geq 3) \\
H^2(P,\mathbb{Z}) &\cong \text{Ext}(H_1(P;\mathbb{Z}),\mathbb{Z}) \oplus \text{Hom}(H_2(P;\mathbb{Z}),\mathbb{Z}) \cong \text{Ext}(\mathbb{Z}_2,\mathbb{Z}) \\
&\cong \mathbb{Z}_2 \\
H^1(P,\mathbb{Z}) &\cong \text{Ext}(H_0(P;\mathbb{Z}),\mathbb{Z}) \oplus \text{Hom}(H_1(P;\mathbb{Z}),\mathbb{Z}) \cong \text{Ext}(\mathbb{Z},\mathbb{Z}) \oplus \text{Hom}(\mathbb{Z}_2,\mathbb{Z}) \\
&\cong 0 \\
H^0(P;\mathbb{Z}) &\cong \text{Hom}(H_0(P;\mathbb{Z}),\mathbb{Z}) \cong \text{Hom}(\mathbb{Z},\mathbb{Z}) \\
&\cong \mathbb{Z}
\end{aligned}$$

Note we can see in this example that $H^n(M;\mathbb{Z})$ need not be zero in a non-orientable manifold. One can also note that the other examples are of orientable manifolds, and note that their cohomology is as dictated by Poincare duality.

Example: The lens space $L = l(p,q)$ with integer coefficients

Recall that

$$\begin{array}{ll}
H_0(L;\mathbb{Z}) \cong \mathbb{Z} & H_1(L;\mathbb{Z}) \cong \mathbb{Z}_p \\
H_2(L;\mathbb{Z}) \cong 0 & H_3(L;\mathbb{Z}) \cong \mathbb{Z} \\
H_q(L;\mathbb{Z}) \cong 0 \text{ for } q > 3 &
\end{array}$$

So

$$\begin{aligned}
H^q(L;\mathbb{Z}) &\cong 0 \text{ for } q > 3 \\
H^3(L;\mathbb{Z}) &\cong \text{Ext}(H_2(L;\mathbb{Z}),\mathbb{Z}) \oplus \text{Hom}(H_3(L;\mathbb{Z}),\mathbb{Z}) \cong \text{Ext}(0,\mathbb{Z}) \oplus \text{Hom}(\mathbb{Z},\mathbb{Z}) \\
&\cong \mathbb{Z} \\
H^2(L;\mathbb{Z}) &\cong \text{Ext}(H_1(L;\mathbb{Z}),\mathbb{Z}) \oplus \text{Hom}(H_2(L;\mathbb{Z}),\mathbb{Z}) \cong \text{Ext}(\mathbb{Z}_p,\mathbb{Z}) \oplus \text{Hom}(0,\mathbb{Z}) \\
&\cong \mathbb{Z}_p \\
H^1(L;\mathbb{Z}) &\cong \text{Ext}(H_0(L;\mathbb{Z}),\mathbb{Z}) \oplus \text{Hom}(H_1(L;\mathbb{Z}),\mathbb{Z}) \cong \text{Ext}(\mathbb{Z},\mathbb{Z}) \oplus \text{Hom}(\mathbb{Z}_p,\mathbb{Z}) \\
&\cong 0 \\
H^0(L;\mathbb{Z}) &\cong \text{Hom}(H_0(L;\mathbb{Z}),\mathbb{Z}) \cong \text{Hom}(\mathbb{Z},\mathbb{Z}) \\
&\cong \mathbb{Z}
\end{aligned}$$

Thus using Ext and the universal coefficient theorem we can compute the cohomology of a wide variety of spaces. Using proposition 7, we can also solve the following application

Proposition 8. *If $H_*^{\#}(Y;F) = 0$ for $F = \mathbb{Q}$, and $F = \mathbb{Z}_p$ for all primes p, then $H_*^{\#}(Y;\mathbb{Z}) = 0$*

Proof. This is clear for H_0. Note that for $q \geq 1$

$$H^q(Y;F) \cong \text{Hom}(H_q(Y;F),F) \cong \text{Hom}(0,F) \cong 0$$

But by the universal coefficient theorem we have

$$H^q(Y;F) \cong \text{Ext}(H_{q-1}(Y;\mathbb{Z}),F) \oplus \text{Hom}(H_q(Y;\mathbb{Z}),F)$$

So $\text{Ext}(H_q(Y;\mathbb{Z}),F) \cong 0$ for all $q \geq 0$, and $\text{Hom}(H_q(Y,\mathbb{Z}),F) \cong 0$ for all $q \geq 1$. In particular $\text{Ext}(H_q(Y,\mathbb{Z}),\mathbb{Z}_p) \cong 0$ for all primes p, so $H_q(Y,\mathbb{Z})$ can have no torsion - if there was $\mathbb{Z}_n$ torsion it would be detected in $\text{Ext}(H_q(Y,\mathbb{Z}),\mathbb{Z}_p)$ for all $p|n$. Likewise as $\text{Hom}(H_q(Y,\mathbb{Z}),\mathbb{Q}) \cong 0$ for $q > 1$, so $H_q(Y,\mathbb{Z})$ has no free part. Thus $H_q(Y,\mathbb{Z}) \cong 0$, and $H_*^{\#}(Y;\mathbb{Z}) \cong 0$ as desired. □

4 The Bockstein Homomorphism

Consider the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_n \to 0$$

Then this short exact sequence induces a short exact sequence of chains on X,

$$0 \to S(X) \otimes \mathbb{Z} \xrightarrow{id \otimes \cdot n} S(X) \otimes \mathbb{Z} \xrightarrow{id \otimes \pi} S(X) \otimes \mathbb{Z}_n \to 0$$

From this we get long exact sequences on both homology and cohomolgy:

$$\xrightarrow{\beta} H_q(X;\mathbb{Z}) \xrightarrow{\eta} H_q(X;\mathbb{Z}) \to H_q(X;\mathbb{Z}_n) \xrightarrow{\beta} H_{q-1}(X;\mathbb{Z}_n) \to$$

and

$$\leftarrow H^{q+1}(X;\mathbb{Z}_n) \xleftarrow{\beta} H^q(X;\mathbb{Z}) \leftarrow H^q(X;\mathbb{Z}) \leftarrow H^q(X;\mathbb{Z}_n) \xleftarrow{\beta}$$

where the connecting homomorphim β is called the Bockstein map, and η is the map induced on homology by multiplication by n. In fact for any short exact sequence of abelian groups

$$0 \to A \to B \to C \to 0$$

we get associated the long exact sequence

$$\xrightarrow{\beta} H_q(X;A) \to H_q(X;B) \to H_q(X;C) \xrightarrow{\beta} H_{q-1}(X;A) \to$$

and likewise the associated sequence on cohomology. However, in some sense the important Bockstein's are those associated with the sequences

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}_n \to 0 \text{ and } 0 \to \mathbb{Z}_n \xrightarrow{\cdot n} \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 0$$

We note the following about the Bockstein associated with $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_n \to 0$. The action of β is defined by the action of the 'snake' map on the ladder of short exact sequences of chain complexes:

$$\begin{array}{ccccccccc} 0 & \to & S_{q-1} \otimes \mathbb{Z} & \xrightarrow{\cdot n} & S_{q-1} \otimes \mathbb{Z} & \xrightarrow{\pi} & S_{q-1} \otimes \mathbb{Z}_n & \to & 0 \\ & & \uparrow \partial & & \uparrow \partial & & \uparrow \partial & & \\ 0 & \to & S_q \otimes \mathbb{Z} & \xrightarrow{\cdot n} & S_q \otimes \mathbb{Z} & \xrightarrow{\pi} & S_q \otimes \mathbb{Z}_n & \to & 0 \end{array}$$

So given a cycle $\sum r_i \sigma_i \otimes \bar{x} \in S_q \otimes \mathbb{Z}_n$, we have that $\{\sum r_i \sigma_i \otimes \bar{1}\} \mapsto \{\frac{1}{n}\partial_q \sum r_i \sigma_i \otimes 1\}$ under β. We note that the Bockstein homomorphism can detect $\mathbb{Z}_n$ torsion.

Proposition 9. *Let Y be a space $H_{q-1}(Y;\mathbb{Z}) = \mathbb{Z}_n$, then $\beta : H_q(Y;\mathbb{Z}_n) \to H_{q-1}(Y;\mathbb{Z})$ is nontrivial.*

Proof. Note that we have the long exact sequence on homology

$$\to H_q(Y;\mathbb{Z}_n) \xrightarrow{\beta} H_{q-1}(Y;\mathbb{Z}) \xrightarrow{\eta} H_{q-1}(Y;\mathbb{Z}) \to$$

We note that η is the map induced by multiplication by n, and as $H_{q-1}(Y;\mathbb{Z}) \cong \mathbb{Z}_n$, if we take any $\tau \in H_{q-1}(Y;\mathbb{Z})$, $n\tau = 0$, so η is the zero map. Exactness implies that β must be onto, and hence non-trivial as desired. □

We conclude with a few examples involving the Bockstein associated with $0 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 0$.

Proposition 10. *$\beta : H^{n-1}(P;\mathbb{Z}_2) \to H^n(P;\mathbb{Z}_2)$ is 0 when n is odd and an isomorphism when n is even, where $P = \mathbb{R}P^n$.*

Proof. Note that $H^q(P;\mathbb{Z}_2) \cong \mathbb{Z}_2$ for $0 \le q \le n$. We compute $H^n(P;\mathbb{Z}_4)$ using the universal coefficient theorem: If n is odd:

$$H^n(P;\mathbb{Z}_4) \cong \mathrm{Ext}(H_{n-1}(P;\mathbb{Z}),\mathbb{Z}_4) \oplus \mathrm{Hom}(H_n(P;\mathbb{Z}),\mathbb{Z}_4) \cong \mathrm{Hom}(\mathbb{Z},\mathbb{Z}_4) \cong \mathbb{Z}_4$$

Hence we have

$$\begin{array}{ccccccccc} H^{n-1}(P;\mathbb{Z}_2) & \xrightarrow{\beta} & H^n(P;\mathbb{Z}_2) & \xrightarrow{\varphi} & H^n(P;\mathbb{Z}_4) & \xrightarrow{\psi} & H^n(P;\mathbb{Z}_4) & \to & 0 \\ \cong \mathbb{Z}_2 & & \cong \mathbb{Z}_2 & & \cong \mathbb{Z}_4 & & \cong \mathbb{Z}_2 & & \end{array}$$

Note ψ is surjective, so im $\varphi = \ker\psi \cong \mathbb{Z}_2$. Thus φ is an isomorphism, and thus injective so β is 0 as desired. If n is even we have

$$H^n(P;\mathbb{Z}_4) \cong \mathrm{Ext}(H_{n-1}(P;\mathbb{Z}),\mathbb{Z}_4) \oplus \mathrm{Hom}(H_n(P;\mathbb{Z}),\mathbb{Z}_4) \cong \mathrm{Ext}(\mathbb{Z}_2,\mathbb{Z}_4) \cong \mathbb{Z}_2$$

and hence we have

$$\begin{array}{ccccccccc} H^{n-1}(P;\mathbb{Z}_2) & \xrightarrow{\beta} & H^n(P;\mathbb{Z}_2) & \xrightarrow{\varphi} & H^n(P;\mathbb{Z}_4) & \xrightarrow{\psi} & H^n(P;\mathbb{Z}_4) & \to & 0 \\ \cong \mathbb{Z}_2 & & \cong \mathbb{Z}_2 & & \cong \mathbb{Z}_2 & & \cong \mathbb{Z}_2 & & \end{array}$$

Note ψ surjective and therefore an isomorphism and injective, so φ is the zero map, and hence β is surjective and therefore an isomorphism as desired. ☐

Proposition 11. *$\beta : H_2(L;\mathbb{Z}_p) \to H_2(L;\mathbb{Z}_p)$ where $L = l(p,q)$, the lens space, is an isomorphism.*

Proof. We begin by computing the homology of the lens space, using the Bockstein associated with $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_n$ Note we have (for any power of p)

$$\begin{array}{ccccccc} H_2(L;\mathbb{Z}) & \to & H_2(L;\mathbb{Z}_p^n) & \xrightarrow{\beta} & H_1(L;\mathbb{Z}) & \xrightarrow{\eta} & H_1(L;\mathbb{Z}) \\ \cong 0 & & & & \cong \mathbb{Z}_p & & \cong \mathbb{Z}_p \end{array}$$

Since η is the map induced by multiplication by p, it is the zero map, and hence we have that $H_2(L;\mathbb{Z}_{p^n}) \cong \mathbb{Z}_p$ for all n, in particular for $n = 1, 2$. We also have

$$H_1^{\#}(L;\mathbb{Z}) \xrightarrow{\eta} H_1^{\#}(L;\mathbb{Z}) \to H_1^{\#}(L;\mathbb{Z}_p) \to 0$$

and as η is the zero map, we have that $H_1(L;\mathbb{Z}_p) \cong H_1(L;\mathbb{Z}) \cong \mathbb{Z}_p$. Also, as by L's construction as a CW complex, we have that $H^3(L;\mathbb{Z}_p) \cong \mathbb{Z}_p$ and $H^3(L;\mathbb{Z}_{p^2}) \cong \mathbb{Z}_{p^2}$. Then we have

$$\begin{array}{ccccccccc} 0 & \to & H_3(L;\mathbb{Z}_p) & \xrightarrow{\eta} & H_3(L;\mathbb{Z}_{p^2}) & \xrightarrow{\varphi} & H_3(L;\mathbb{Z}_p) & \xrightarrow{\beta} & H_2(L;\mathbb{Z}_p) \\ & & \cong \mathbb{Z}_p & & \cong \mathbb{Z}_{p^2} & & \cong \mathbb{Z}_p & & \cong \mathbb{Z}_p \end{array}$$

Note that $\ker\varphi \cong$ im $\eta \cong \mathbb{Z}_p$, so φ is surjective, and hence β is the zero map, so further down we have

$$\begin{array}{ccccccccc} H_3(L;\mathbb{Z}_p) & \xrightarrow{0} & H_2(L;\mathbb{Z}_p) & \xrightarrow{\eta} & H_2(L;\mathbb{Z}_{p^2}) & \xrightarrow{\psi} & H_2(L;\mathbb{Z}_p) & \xrightarrow{\beta} & H_1(L;\mathbb{Z}_p) \\ \cong \mathbb{Z}_p & & \cong \mathbb{Z}_p & & \cong \mathbb{Z}_p & & \cong \mathbb{Z}_p & & \cong \mathbb{Z}_p \end{array}$$

Note that η is injective, and thus surjective, so ψ is the zero map, and hence β is injective and hence an isomorphism as desired (as all groups are $\cong \mathbb{Z}_p$). ☐