290 Review part 14

Definition of cup product: The cup product is a bilinear map $\cup : S^p(X) \times S^q(X) \to S^{p+q}(X)$. If $c \in S^p(X)$ and $d \in S^q(X)$, we can define the value of $c \cup d$ on every (p+q)-simplex and extend by linearity to define $c \cup d$ on every element of S_{p+q}. Therefore, let $\sigma \in S_{p+q}$. σ is a map $\triangle_{p+q} \to X$. Suppose $\triangle_{p+q}$ has vertices $v_0, v_2, \cdots v_{p+q}$. We define the value of $c \cup d$ on σ to be the value of c on σ restricted to the first p variables times the value of d on σ restricted to the last q variables, i.e. $[\sigma, c \cup d] = [\sigma|(v_0, ..., v_p), c][\sigma|(v_p, ..., v_{p+q}), d]$.

Once we have defined the cup product for $S^p(X) \times S^q(X)$ we can extend to a map $\cup : S^\bullet(X) \times S^\bullet(X) \to S^\bullet(X)$, where $S^\bullet(X) = \bigoplus S^q(X)$. If $c = \sum c_p, d = \sum d_q$ then $c \cup d = \sum c_p \cup d_q$.

In order to pass to cohomology by quotient we must look at how the coboundary interacts with the cup product. We have the formula $\delta(c \cup d) = \delta c \cup d + (-1)^p c \cup \delta d$ for $c \in S^p(X)$ and $d \in S^q(X)$. Therefore the cup product of two cocycles is a cocycle. Also if $\delta c = 0$ (c is a cycle), we have $c \cup \delta d = \pm\delta(c \cup d)$. And similarly if $\delta d = 0$ (d is a cycle) we have $\delta c \cup d = \delta(c \cup d)$. Therefore the cup product of a cocycle with a coboundary is a coboundary. This gives that $Z^\bullet(X)$ is a subring of $S^\bullet(X)$ and $B^\bullet(X)$ is a two sided ideal of $Z^\bullet(X)$. Hence we can pass by quotient to cohomlogy $H^\bullet(X)$.

Definition of cap product: The cap product is a bilinear map $\cap : S_{p+q}(X) \times S^p(X) \to S_q(X)$. Given $\sigma \in S_{p+q}(X)$ and $c \in S^p(X)$, $\sigma \cap c = [\sigma|(v_0, ..., v_p), c]\sigma|(v_p, ..., v_{p+q})$. We extend to (p+q)-chains by linearity. Note that if $z \in S_{p+q}(X)$, $z \cap c$ is the unique q-chain such that $[z \cap c, d] = [z, c \cup d]$ for all $d \in S^q(X)$.

As in the cup product, we can extend to a map $\cap : S_\bullet(X) \times S^\bullet(X) \to S_\bullet(X)$ by linearity. The formula $\partial(z \cap c) = (-1)^p[(\partial z) \cap c - z \cap \delta c]$ shows that a cycle capped with a cocycle is a cycle. And if z is a cycle $\partial(z \cap c) = \pm(z \cap \delta c)$, so a cycle cap a coboundary is a boundary. If c is a cocycle $\partial(z \cap c) = \pm((\partial z) \cap c)$, so a boundary cap a cocycle is a boundary. Therefore we can pass to the quotient $\cap : H_{p+q}(X) \times H^p(X) \to H_q(X)$.

Orientation Class of T: (Also called the fundamental class of T.) An R-orientation of T is determined by a generator of $H_2(T)$. This generator is the orientation class of T.

(this will be a picture of a square with vertices labeled A_0, A_1, B_0, and B_1 and opposite sides identified).

We let ϕ be the map that identifies the edges in the diagram. We will show that $z = \phi(A_0A_1B_1) - \phi(A_0B_0B_1)$ is such a cycle. First we show that z is a cycle. $\delta z = \phi(A_1B_1) - \phi(A_0B_1) + \phi(A_0A_1) - \phi(B_0B_1) + \phi(A_0B_1) - \phi(A_0B_0) = 0$.

From our original calculation of $H_2(T)$ we have the isomorphisms $H_2(T) \to H_2(T, S^1 \bigvee S^1) \to H_2(E^2, S^1) \to H_1(S^1)$. Thus we have $\{z\}$ maps to $(A_0A_1) + (A_1B_1) - (B_0B_1) - (A_0B_0)$ which generates $H_1(S^1)$. Hence $\{z\}$ is a generator of $H_2(T)$.

Cup Product Structure of T: Assuming R is a PID, $H^1(T) = R \oplus R$ (since $H^1(T) = Hom(H_1(T; \mathbb{Z}, R))$. The generators of $H_1(T)$ are α and β where α is the homology class of the loop $\phi(A_0A_1)$ and β is the homology class of the loop $\phi(A_0B_0)$. Since any element of $H^1(T)$ is determined by it's values at α and β, the generators of $H^1(T)$ are α^* and β^* where $[\alpha, \alpha^*] = 1$, $[\beta, \alpha^*] = 0$, $[\beta, \beta^*] = 1$, and $[\alpha, \beta^*] = 0$.

We will show that $\alpha^* \cup \beta^*$ generates $H^2(T)$. $H^2(T)$ is isomorphic to R by sending any element of $H^2(T)$ to the value it takes on z, the generator of $H_2(T)$. Therefore, we need to check that $[z, \alpha^* \cup \beta^*] = 1$.

$[z, \alpha^* \cup \beta^*] = [\phi(A_0A_1B_1), \alpha^* \cup \beta^*] - [\phi(A_0B_0B_1), \alpha^* \cup \beta^*] = [\phi(A_0A_1), \alpha^*][\phi(A_1B_1), \beta^*] - [\phi(A_0B_0), \alpha^*][\phi(B_0B_1), \beta^*]$ from the definition of cup product. Since $\phi(A_1B_1)$ is the same loop as $\phi(A_0B_0)$ and $\phi(B_0B_1)$ is the same loop as $\phi(A_0A_1)$ we have $[\alpha, \alpha^*][\beta, \beta^*] - [\beta, \alpha^*][\alpha, \beta^*] = 1 + 0 = 1$. Therefore $[z, \alpha^* \cup \beta^*] = 1$ and $\alpha^* \cup \beta^*$ generates $H^2(T)$. Thus we know that α^* and β^* generate $H^\bullet(T)$, so it only remains to find the relations on α^* and β^* to determine the cup product structure.

$[z, \alpha^* \cup \alpha^*] = [\alpha, \alpha^*][\beta, \alpha^*] - [\beta, \alpha^*][\alpha, \alpha^*] = 0$

$[z, \beta^* \cup \beta^*] = [\alpha, \beta^*][\beta, \beta^*] - [\beta, \beta^*][\alpha, \beta^*] = 0$

$\alpha^* \cup \beta^* = -\beta^* \cup \alpha^*$ since $\alpha^* \cup \beta^* = (-1)^{pq}\beta^* \cup \alpha^*$

Cap Product Structure:

$z \cap \alpha^* = [\phi(A_0A_1), \alpha^*]\phi(A_1B_1) - [\phi(A_0B_0), \alpha^*]\phi(B_0B_1) = \phi(A_1B_1) - 0 = \beta$

$z \cap \beta^* = [\phi(A_0A_1), \beta^*]\phi(A_1B_1) - [\phi(A_0B_0), \beta^*]\phi(B_0B_1) = 0 - \phi(B_0B_1) = -\alpha$

Now we can use the cup product structure to distinguish between two spaces with the same homology. If $f : T \to S^1 \vee S^1 \vee S^2$ were a homotopy equivalence then $H^\bullet(f) : H^\bullet(S^1 \vee S^1 \vee S^2) \to H^\bullet(T)$ would be an isomorphism. However since $H^1(S^1 \vee S^1 \vee S^2)$ is equal to $H^1(S^1 \vee S^1)$, the cup product of the two generators of $H^1(S^1 \vee S^1 \vee S^2)$ is 0, not the generator of $H^2(S^1 \vee S^1 \vee S^2)$. This contradicts the fact that $H^\bullet(f)$ is a ring homomorphism.