15: FUNDAMENTAL GROUP

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As in lecture, assume every space we deal with is connected, path connected, locally path connected, and semi-locally simply connected. If a proof requires one of these, it need not always be an explicit hypothesis. Way back at the beginning, we defined a loop σ as some continous map from the unit interval I to a space X so that $\sigma(0) = \sigma(1)$. For convenience, we can also think of loops as maps $S^1 \to X$; the definitions are equivalent. A loop at x_0, where x_0 is the basepoint, simply means such a map where $\sigma(0) = x_0$. Loops turn out to be useful in detecting 'holes' in topological spaces. If we considered all possible loops at some basepoint, we would have some crazy uncountable set. Instead, we identify loops that are homotopic relative $\{0, 1\}$.

We define $\pi_1(X, x_0)$ as the set of (relative $\{0, 1\}$) homotopy classes of loops in X with basepoint x_0. We denote some element by $[\sigma]$, which means the (equivalence) class of σ. Since we learned how to multiply loops (any paths, in fact), we find that $\pi_1(X, x_0)$ is a group (check chapter 2 for details: the important point to remember is that $\sigma\tau$ means first go around σ, then go around τ). π_1 is called the fundamental group.
It is critical that we care about homotopies relative $(0, 1)$. This means that we must keep the basepoint fixed when homotoping one loop into another. This potentially gives us fewer equivalence classes than we would have looking at all homotopy classes. When looking at something simple like the circle, we tend to forget the relative $(0, 1)$ condition because loops are homotopic iff they're homotopic relative $(0, 1)$. Moreover, we showed in an exercise that if every loop in a space X is homotopically trivial (not necessarily relative $(0, 1)$) then $\pi_1(X, x_0) = 0$. So why is that we care so much about this relative homotopy business? Well, it starts to make a difference as soon as we add another layer of complication. Recall that $\pi_1(S^1 \vee S^1, x_0) \cong \mathbb{Z} * \mathbb{Z}$, the non-abelian free product of $\mathbb{Z}$ with itself. Let σ be a loop wrapping once around one circle and τ a loop wrapping once about the other, both with basepoint x_0. Then $\sigma\tau$ is not homotopic to $\tau\sigma$ rel$(0, 1)$. However,
Claim: $\sigma\tau \simeq \tau\sigma$.
Define $F : I \times I \to S^1 \vee S^1$ by

$$F(s,t) = \begin{cases} \sigma(2s+t) & 0 \le s < \frac{1-t}{2} \\ \tau(2s+t-1) & \frac{1-t}{2} \le s < \frac{2-t}{2} \\ \sigma(2s+t-2) & \frac{2-t}{2} \le s \le 1 \end{cases}$$

Let's check that F provides our homotopy. First we need to verify continuity. By the pasting lemma, we need only check that our functions agree at the endpoints. $\sigma(2\frac{1-t}{2} + t) = \sigma(1) = x_0$ and $\tau(2\frac{1-t}{2} + t - 1) = \tau(0) = x_0$, so the first junction works.
$\tau(2\frac{2-t}{2} + t - 1) = \tau(2-1) = \tau(1) = x_0$ and $\sigma(2\frac{2-t}{2} + t - 2) = \sigma(2-2) = \sigma(0) = x_0$, so the 2nd does as well. Now we check that it evaluates to the right things

$$F(s,0) = \begin{cases} \sigma(2s) & 0 \le s < 1/2 \\ \tau(2s-1) & 1/2 \le s < 1 \\ \sigma(2s-2) & 1 \le s \le 1 \end{cases} = \quad \begin{cases} \sigma(2s) & 0 \le s < 1/2 \\ \tau(2s-1) & 1/2 \le s < 1 \end{cases} = \quad (\sigma\tau)(s)$$

$$F(s,1)=\begin{cases}\sigma(2s) & 0\le s<0\\ \tau(2s-1) & 0\le s<1/2\\ \sigma(2s-2) & 1/2\le s\le 1\end{cases}=\quad\begin{cases}\tau(2s-1) & 0\le s<1/2\\ \sigma(2s-2) & 1/2\le s\le 1\end{cases}=\quad(\tau\sigma)(s)$$

Thus F provides the desired homotopy. Of course, $F(0,t)\neq F(1,t)\neq x_0$ (except for a couple values of t). Thus the homtopy is not relative $(0,1)$.
In contrast to this result, we had to pick a basepoint when working with the fundamental group, but we found that the choice of basepoint did not usually matter. Stating this more precisely:

Proposition: *If X is path connected, $\pi_1(X,x_0)$ is independent of x_0.*

Proof. To see this, we pick any distinct points $x_0,x_1\in X$ and need to show
$\pi_1(X,x_0)\cong\pi_1(X,x_1)$.
The way to do this is relatively obvious: pick some path α from x_0 to x_1, which is possible by assumption. As with most proofs where you choose the right paths, the basic idea is easy but checking the details takes time.
Claim: the map $\alpha_*:\pi_1(X,x_0)\longrightarrow\pi_1(X,x_1)$ sending $[\sigma]$ to $[\alpha^{-1}\sigma\alpha]$ is an isomorphism.
We check this is well-defined. Say $\sigma_1\simeq\sigma_2$ rel$(0,1)$, so we have some $F_t:I\to X$ $(t\in[0,1])$ s.t. $F_0=\sigma_1, F_1=\sigma_2$, $F_t(0)=F_t(1)=x_0$.
We need to show $\alpha^{-1}\sigma_1\alpha\simeq\alpha^{-1}\sigma_2\alpha$ rel$(0,1)$.
Claim: $G_t=\alpha^{-1}F_t\alpha$ works as our desired homotopy. As the composition of continuous functions, it is continuous.
$G_0=\alpha^{-1}F_0\alpha=\alpha^{-1}\sigma_1\alpha$.
$G_1=\alpha^{-1}F_1\alpha\alpha^{-1}\sigma_2\alpha$
$G_t(0)=G_t(1)=(\alpha^{-1}x_0\alpha)(0)=x_1$
(this loop first moves along α^{-1} from x_1 to x_0, then sits at x_0, then moves from x_0 to x_1, so the starting/ending point is x_1).
Now we check that α_* is a homomorphism:
$\alpha_*([\sigma_1][\sigma_2])=\alpha_*([\sigma_1\sigma_2])=[\alpha^{-1}\sigma_1\sigma_2\alpha]=[\alpha^{-1}\sigma_1(\alpha\alpha^{-1})\sigma_2\alpha]=[\alpha^{-1}\sigma_1\alpha][\alpha^{-1}\sigma_2\alpha]=\alpha_*(\sigma_1)\alpha_*(\sigma_2)$.
Next check that it is injective. Suppose $[\alpha^{-1}\sigma\alpha]=0$, so that $\alpha^{-1}\sigma\alpha\simeq x_1$, the constant loop. Then we have a homotopy F_t between them and $\alpha F_t\alpha^{-1}$ provides a homotopy between σ and $\alpha x_1\alpha^{-1}=x_0$. Just follow the same process as for well-definedness. So $[\sigma]=0$.
Finally, if we get surjectivity we're done. Pick some $[\tau]\in\pi_1(X,x_1)$. Claim: $\alpha_*(\alpha\tau\alpha^{-1})=\tau$. This is pretty clear by definition; the only question is checking that $\alpha\tau\alpha^{-1}$ is a loop at x_0 It starts at x_0, traces a path to x_1, runs around the loop τ around x_1, then traces along the path back to x_0. Thus it is a loop at x_0. □

Note: Recall that π_1 (with basepoint) is a functor.
Definition: Simply connected = path connected with trivial fundamental group.
Recall that we found the fundamental group of the circle by looking at the circle as $\mathbb{R}/\mathbb{Z}$. The generalization of this method is:
Theorem 4.5 *If G is a simply connected topological group and H a discrete normal subgroup of G, then $\pi_1(G/H,1)\cong H$.*

Proof. Greenberg and Harper achieve this result via liftings as with $\mathbb{R}/\mathbb{Z}$ and checking an extra detail. Now that we have more tools available, we can use covering spaces and some algebra.
Claim: G covers G/H via the quotient map p.
For notation, I'll use $p(g)=[g]$, the equivalence class of g, where $[g_1]=[g_2]$ iff $g_1\in Hg_2$.
Hg means $\{hg:h\in H\}$.
Since p defines the quotient topology on G/H, we automatically get that p is continuous and onto. Also, since H is a discrete subgroup, our fibers $p^{-1}([g])$ are all discrete. Thus all we really need to check is that it's an even covering (check Oded's writeup if you need refreshing

on coverings).
Pick some open $U \subset G/H$ containing a point $[g]$. Then $p^{-1}(U)$ gives us neighborhoods of a bunch of $g_\alpha \in G$, where $[g_\alpha] = [g]$. If we pick any set of representative elements $\{u_\beta\}$ for for our equivalence classes in U (possibly using axiom of choice), we get neighborhoods in G of the form $V_\alpha = h_\alpha\{u_\beta\}$. We already have $p|_{V_\alpha}$ onto and continuous. We can let $p^{-1}([u_\beta]) = h_\alpha u_\beta$, so we get a continous, onto inverse and an overall homeomorphism.
Now we can use thm 5.8, which says that $\pi_1(G/H, 1)$ is isomorphic to the group of covering transformations of G. It remains only to check that H gives us all covering transformations. Pick any covering transformation ϕ. Then we have $p\phi = p$, so we get the diagram

$$\begin{tikzcd}
G, e \arrow[rr, "\phi"] \arrow[dr, "p"] & & G, \phi(e) \arrow[dl, "p"'] \\
& G/H, [e] &
\end{tikzcd}$$

Let $h = \phi(e)$, which must be in H since $p^{-1}([e]) = H$.
Let β be a path from e to h. Then $\phi \circ \beta$ is a path from h to $\phi(h)$.
Since $p(h) = p\phi(h)$, $p\phi\beta = p\beta$ is a loop in G/H. $\phi \circ \beta$ is clearly a lifting of this loop. Since $p(h \cdot \beta(t)) = [h \cdot \beta(t)] = [\beta(t)]$, the path $t \mapsto h \cdot \beta(t)$ is also a lifting of $p\beta$. Moreover, $h \cdot \beta(0) = \phi \circ \beta(0) = h$. But we know that if 2 liftings of a path agree at a point, they must be equal. So we have $\phi\beta(t) = h \cdot \beta(t)$ for all $t \in [0, 1]$. Since G is path connected, we can have β pass through any point. Thus we have $\phi(g) = h \cdot g$ for all $g \in G$. We see that our covering transformations are exactly multiplication by elements of H, so the result follows. ☐

Proposition 4.8 $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$ See Greenberg and Harper for the proof. From this or above, we get the fundamental group of the torus is $\mathbb{Z} \times \mathbb{Z}$.

Thm *If G is a topological group with identity e, $\pi_1(G, e)$ is abelian.*

Proof. The idea here is that we can define another type of multiplication. Normally, $\sigma\tau$ means go around σ twice as fast as normal, then go around τ twice as fast. We now define $\sigma * \tau$ to be multiplication in the group: For any $t \in I$, $(\sigma * \tau)(t) = \sigma(t) \cdot \tau(t)$.
Since each piece is continuous in t and multiplication by a group element is continuous (in fact, it's a homeomorphism), $\sigma * \tau$ is indeed a continuous map $I, \{0, 1\} \to G, e$, so it is a new loop.
Now we look at a couple homotopies:
$F(s, t) = \sigma(s)\tau(st)$ represented by

$$\begin{tikzcd}
\cdot \arrow[r, "\sigma * \tau"] & \cdot \\
\cdot \arrow[u, "e"] \arrow[r, "\sigma"] & \cdot \arrow[u, "\tau"']
\end{tikzcd}$$

$G(s,t) = \sigma(st)\tau(st)$ represented by

$$\begin{tikzcd}
\cdot \arrow[r, "\sigma * \tau"] & \cdot \\
\cdot \arrow[u, "e"] \arrow[r, "\tau"] & \cdot \arrow[u, "\sigma"']
\end{tikzcd}$$

Using lemma 3.3, which says these diagrams commute, we get
$\sigma * \tau \simeq e^{-1}\sigma\tau \ \mathrm{rel}(0,1) = \sigma\tau$ (since e is the identity) and
$\sigma * \tau \simeq e^{-1}\tau\sigma \ \mathrm{rel}(0,1) = \tau\sigma$.
By transitivity, we have $\sigma\tau \simeq \tau\sigma \ \mathrm{rel}(0,1)$, so $\pi_1(G,e)$ is commutative. □

Exercise 4.12 Show that if $X = U \cup V$ with U and V simply connected, and $U \cap V \neq \emptyset$ is path connected, then X is simply connected.

Proof. Let $f \in \pi_1(X, x_0)$. Since we have shown that $\pi_1(X)$ is independent of basepoint, we can assume $x_0 \in U \cap V$.
Step 1: We want to decompose the loop f into a finite number of paths, each in either U or V. Let $\{I_\alpha\}$ be the set of open intervals of $[0,1]$ that map to either U or V, but not both.
Claim: $\bigcup_\alpha I_\alpha = [0,1]$.
Pick any $s \in [0,1]$. WLOG, say $f(s) \in U$. Since U is open, it contains some neighborhood of $f(s)$, and by continuity of f we can find some interval (a,b) around s that is mapped to this neighborhood. Thus $f((a,b)) \subset U$, so $(a,b) \in \{I_\alpha\}$. Thus $s \in \bigcup_\alpha I_\alpha$.
Since $[0,1]$ is compact, there is a finite subcover $I_1, \dots, I_n$. Without loss of generality, if two intersecting I_j, I_k both map under f to the same space (U or V), we replace them with their union, so adjacent intervals map to different spaces. By doing this, we can order our intervals, which intersect only with their adjacent neighbors, by left endpoint.
Now we can choose $s_i \in I_i \cap I_{i+1}$, setting $s_0 = 0$ and $s_n = 1$.
Then we thus have closed intervals $[s_0, s_1], [s_1, s_2] \dots [s_{n-1}, s_n]$.
Let f_j be the path $f([s_{j-1}, s_j])$, which means:

$$\begin{array}{rcl} f_n : I & \to & X \\ t & \mapsto & f(s_{j-1}(1-t) + s_j t) \end{array}$$

Note that our choice of s_j required $s_j \in U \cap V$ for each j (since I_i and I_{j+1} will map to different spaces). Thus we have s_j and x_0 in $U \cap V$, which is path connected. So we can let g_j be a path in $U \cap V$ from x_0 to s_j. Then $g_j^{-1} g_j \simeq a_j \ \mathrm{rel}(0,1)$ since both are loops in a simply connected and therefore contractible space. So $f \simeq (g_0 f_1 g_1^{-1})(g_1 f_2 g_2^{-1}) \dots (g_{n-1} f_n g_n)\mathrm{rel}(0,1)$
Claim: each term in parentheses is a loop in either U or V. By construction, it lies all in U or all in V. As a composition of continous paths, each is continuous. Also $g_{j-1}^{-1} f_j g_j(0) = g_j(0) = x_0$; $g_{i-1}^{-1} f_i g_i(1) = g_{i-1}^{-1}(1) = x_0$, so it's a loop.
Since U and V are simply connected, each loop $g_{j-1}^{-1} f_j g_j$ is homotopic to the trivial loop relative $(0,1)$. By transitivity of homotopy, we get $[f] = (x_0) \dots (x_0) = x_0$ as desired. □

This is a particular case of the Van Kampen theorem, which says that for $X = U \cup V$, $U, V, U \cap V$ non-empty and path-connected, $\pi_1(X)$ is given by the amalgamated free product of $\pi_1(U)$ and $\pi_1(V)$. What exactly does amalgamated mean, you might ask. Well, we get a homomorphism induced by inclusions: $\Phi : \pi_1(U) * \pi_1(V) \to \pi_1(X)$ that is surjective, so $\pi_1(X) \cong \frac{\pi_1(U) * \pi_1(V)}{\ker \Phi}$. Moreover, $\ker \Phi$ is the (normal) subgroup generated by elements of the form $i_U(\sigma) i_V(\sigma)^{-1}$, where i_U and i_V are induced by inclusions: $i_U : \pi_1(U \cap V) \to \pi_1(U)$,

$i_V : \pi_1(U \cap V) \to \pi_1(V)$ and $\sigma \in \pi_1(U \cap V)$. In particular, if $U \cap V$ is simply connected, we just get $\pi_1(X) \cong \pi_1(U) * \pi_1(V)$. See Hatcher section 1.2 for details/proof.

Example $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$.

We let U and V be the respective circles plus a little extra so that they are open. Then $U \cap V$ will be an 'x' containing the basepoint, which is contractible so simply connected. Moreover, S^1 is a deformation retract of both U and V. Thus $\pi_1(S^1 \vee S^1) \cong \pi_1(U) * \pi_1(V) \cong \pi_1(S^1) * \pi_1(S^1) \cong \mathbb{Z} * \mathbb{Z}$. More generally, as long as we have what we can now call collared pairs $(X, X \vee Y)$ and $(Y, X \vee Y$, we will get $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$.

Corollary $S^1 \vee S^1$ *does not have the homotopy type of a topological group.* Suppose it did. Then $\pi_1(S^1 \vee S^1, x_0)$ would be abelian. But the free product on 2 copies of $\mathbb{Z}$ is not.

We might try to use a universal covering space of $S^1 \vee S^1$ to get the same results. Hatcher has a lot of discussion (and examples) about the different covering spaces of it on pages 57-59. As better be true since $S^1 \vee S^1$ is sufficiently connected, there is a universal cover. It is a sort of infinite snowflake of crosses (pictured in Hatcher). Unfortunately, figuring out the group of covering transformations is not so easy now. I can see a few obvious ones: the identity, flipping about either or both axes. In this case, we would rather go the other direction: because we've shown $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$, it must be that the set of covering transformations of this universal cover is given by $\mathbb{Z} * \mathbb{Z}$. In particular, there must be at least that many homeomorphisms of the space into itself.

Proposition *Given any finitely generated group G, we can construct a space X for which $\pi_1(X, x_0) = G$.*

Proof. If we were only looking at abelian groups, we could follow the same process as constructing a space with any homology. Instead, we first need to remind ourselves how to characterize any finitely generated group. We can present it as a set of generators with relations:
$\langle a_1, \dots, a_n : r_1, \dots, r_k \rangle$, where the relations r_k are 'words' that are set to 1–for instance, $r_j = a_4 a_7^{-1} a_5$ means we set $a_4 a_7^{-1} a_5 = 1$.
If we have n generators, we start with the wedge of n circles $S^1 \vee \cdots \vee S^1$ (all with the same basepoint: a rose rather than a chain). By repeated applications of Van-Kampen, we know the fundamental group of this will be free on n generators, where we can say a_1 goes around the first circle once, a_2 goes around the second once, and so on. Now we need to take care of the relations. We do this by attaching a 2-cell via the map $f_j : S^1 \to S^1 \vee \cdots \vee S^1$ that follows the path r_j. So again, if $r_j = a_4 a_7^{-1} a_5$, our map f would go around the 4th circle clockwise, then the 7th circle counterclockwise, then around the 5th circle clockwise. We connect disks via these maps for each j up to k, and then we should be done. If we view r_j as a loop, we have just attached a disk along this loop, so the loop can now be shrunk to a point within that disk, which is contained in $S^1 \vee \cdots \vee S^1 \cup_{f_j} E^2$. Thus it will be homotopically trivial. Thus we get each $r_j = 1$ in $S^1 \vee \cdots \vee S^1 \cup_{f_1} E^2 \cdots \cup_{f_k} E^2$. Not only do we have our result, but we achieved it with a 2-dimensional cell complex. □

Exercise 4.13 S^n *is simply connected for* $n \geq 2$

Proof. This can be done as a corollary to 4.12 above. Let U be S^n minus the north pole, V be S^n minus the south pole. Then both U and V are simply connected (each is homeomorphic to $\mathbb{R}^n$). When $n \geq 2$, $U \cap V$ will also be path connected. Thus we get S^n simply connected.
If you prefer a more hands-on approach, read on.
Basically, we take a typical loop (i.e. a circle) and slide it around over whichever pole, keeping the basepoint fixed, until we bring the rest of the loop around and move it to the basepoint. Unfortunately, things become more difficult when we consider all kinds of loops.
Claim: Any non-surjective loop is nullhomotopic (note: this applies for any n).

pf: Let σ be a non-surjective loop, let $x \in S^n$ be a point not hit by σ. Then σ factors through $S^n - \{x\}$, which is contractible, so σ can be shrunk to a point in $S^n - \{x\}$. The same homotopy will then shrink it in S^n, so the σ is nullhomotopic.
Now we need to show that for $n \geq 2$, any loop in S^n is homotopic to a non-surjective one. We do this by picking some particular point and moving over all parts of the loop that hit that point. Let σ be a surjective loop. Pick some open ball B homeomorphic to $\mathbb{R}^n$ around a point $x \in S^n$. Since σ is cont, $\sigma^{-1}(B)$ is open in $[0,1]$, so it can be expressed as a disjoint union of open intervals. $\{x\}$ is closed in S^n, so it is compact, thus $\sigma^{-1}(\{x\})$ is compact. Our disjoint union of open intervals for $\sigma^{-1}(B)$ covers $\sigma^{-1}(\{x\})$, so there must be a finite subcover $(a_1, b_1) \ldots (a_n, b_n) \subset [0,1]$. Since $\sigma(a_i, b_i) \subset B$, $\sigma([a_i, b_i]) \subset \overline{B}$ with $\sigma(a_i)$ and $\sigma(b_i)$ on the boundary of B. This is finally the part where we use the dimension. For $n = 1$, there is no way to avoid hitting x. However, if $n \geq 2$, then the boundary of B (a sphere of dimension $n-1$) will be path connected. Let f_i be a path along the boundary of B from $\sigma(a_i)$ to $\sigma(b_i)$. Now define $\tau_0 = \sigma$ and Let $\tau_i(s) = \begin{cases} \tau_{i-1}s & s \notin [a_i, b_i] \\ f_i(s) & s \in [a_i, b_i] \end{cases}$
By our definition of f_i, this is a continuous map (2 continuous pieces agree at a_i, b_i), so each τ_i is still a loop.
By pushing all points in $\tau_{i-1}(a_i, b_i)$ radially outward toward the boundary of B, we get a homotopy so that $\tau_{i-1} \simeq \tau_i$ rel$(0,1)$. Using transitivity and a finite number of steps, we thus achieve $\sigma \simeq \tau_n$rel$(0,1)$. By design, τ_n never hits the point x, so we're done. □

Example $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}_2$ for $n \geq 2$.
Now that we know $\pi_1(S^n) = 0$ for $n \geq 2$, it provides a simply connected cover for $\mathbb{R}P^n$. We use covering space theory, as in thm 4.5, to say the fundamental group of projective space is isomorphic to the group of covering transformations of S^n. The only covering transformations are the identity and the antipodal map, and any group of 2 elements is isomorphic to $\mathbb{Z}_2$, so $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}_2$.

Exercise 4.14 *Show that every* 3×3 *matrix with positive real entries has an eigenvector with positive eigenvalue.*

Proof. There are different ways to do this, all of which amounting to showing that the matrix provides a map from something homeomorphic to E^2 into itself.
Let A be a 3×3 matrix with positive real entries. Define a linear transformation on $\mathbb{R}^3$ by $T(x) = \frac{Ax}{\|Ax\|}$. Restricting to the set $V = \{x : x_i \geq 0, \|x_i\| = 1\}$ (the unit ball intersect the positive octant), we get that $Ax \neq 0$, so this map is continuous. V is homeomorphic to E^2 (in the same way the upper hemisphere is), so T induces a continuous map $E^2 \to E^2$. The Brouwer fixed point theorem says any such map has a fixed point. Thus there must be some y for which $T(y) = y$. Then $\frac{Ay}{\|Ay\|} = y$ or $Ay = \|Ay\|y$. Thus $\|Ay\|$ is a positive eigenvalue for the eigenvector y. □