Topology Writeup # 19

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1. Cohomology with Compact Support: *Fun with Direct Limits*

So. Why is it necessary to come up with a new version of cohomology which is more contrived than the standard singular theory? After all, if our goal is the Poincaré Duality theorem, which asserts roughly that for compact, R-orientable n-manifolds M, the map $\zeta_M \cap : H^q(M;R) \to H_{n-q}(M;R)$ is an isomorphism (here ζ_M denotes a fundamental class of M), where do these more bizarre cohomology modules rear their ugly heads? A first observation would be that if one were to remove compactness from the above statement, certain key parts break. Most notably, the existence of a fundamental class ζ_M is no longer guaranteed in a non-compact manifold ($\mathbb{R}^n$, for example). Added to this is the fact that such a duality theorem, even if one could define a suitable map, is still not true (again, look at the homology and cohomology of $\mathbb{R}^n$). But what of it? Well, the reason we still require this machinery is that to prove Poincaré duality we'll appeal to the time-honored method for proving facts about manifolds: first prove the result in $\mathbb{R}^n$ and then "patch together" charts on the manifold (in our setting using a Meyer-Vietoris argument) to obtain the global result. So, at the end of the of the day, we'll need to work on $\mathbb{R}^n$ anyway, and thus we'll need a setting in which duality holds. So, let's get cracking. . .

1.1. Direct Limits.

We'll first need to work through this purely algebraic material. We begin with defining the indexing sets over which limits may be taken.

Definition 1.1. *A set I is said to be* directed *or* filtering *if it has a partial order $\leq$ with the property that for any $i_1, i_2 \in I$ there is an i_3 with $i_1 \leq i_3$ and $i_2 \leq i_3$.*

One can provide many examples of such sets, but really the one we will be concerned about is the example of set inclusion, namely, if X is a set and X_1 and X_2 are subsets, then $X_1 \subset X_1 \cup X_2$ and $X_2 \subset X_1 \cup X_2$. So all subsets of X under the partial order $\subset$ is a directed set. We'll use this example in our construction of cohomology with compact support, with the added restriction that the subsets be compact.

Now for the somewhat painful definition of the direct limit:

Definition 1.2. *Given a set of modules over R, $\{M_i\}_{i \in I}$, indexed by I a directed set, with the additional property that for each relation $i \leq j$ there is a homomorphism $\phi_{ji} : M_i \to M_j$ that satisfies:*

(1) *ϕ_{ji} is the identity if $i = j$,*
(2) *$\phi_{kj} \circ \phi_{ji} = \phi_{ki}$ if $i \leq j \leq k$,*

then we say this is a direct *or* inductive *system of modules.*

Given a direct system of modules, we can form a direct or inductive limit *of the system. This consists of a module M and a family of homomorphisms $\phi : M_i \to M$, also indexed by I, with the* *property that* $\phi_j \circ \phi_{ji} = \phi_i$ *if* $i \leq j$*, and which satisfies a so-called* universal *property: For any module* M' *equipped with a family of homomorphisms* $\phi' : M'_i \to M'$ *such that* $\phi'_j \circ \phi_{ji} = \phi'_i$ *if* $i \leq j$ *there is unique homomorphism* $\phi' : M \to M'$ *such that* $\phi'_i = \phi \circ \phi_i$ *for each* i.

If such a limit exists, we write $\varinjlim M_i = M$. The unique map ϕ' can be denoted $\varinjlim \phi'_i$. It is useful to note that if two such limits exist, they're isomorphic (hence, talking about "the" direct limit is kosher). I could give this a clearly, but instead I'll show why this is true so we can start playing with these gadgets.

Proposition 1.3. *Any two direct limits are isomorphic.*

Proof. Say M and M' are both direct limits of the system of M_i's. Then let $\psi_1 : M \to M'$ and $\psi_2 : M' \to M$ be the maps guaranteed by the universal property. Thus, $\psi_1 \circ \psi_2 : M' \to M'$ and $\psi_2 \circ \psi_1 : M \to M$ are both the identity, buy the uniqueness condition of the universal property. Thus, our conclusion. □

So we know that if limits exist, they're unique (up to isomorphism). But do they exist? In fact, yes, they always exist. The construction is to take the direct sum $\bigoplus_{i \in I} M_i$ combined with maps $\phi_i^+ : M_i \to \bigoplus_{i \in I} M_i$ which sends each $m_i \in M_i$ to the element with m_i in the i^{th} component and 0 otherwise. We then quotient out by the submodule generated by elements of the form

$$\phi_j^+ \circ \phi_{ji}(m_i) - \phi_i^+(m_i)$$

whenever $i \leq j$ and for any $m_i \in M_i$. The resulting quotient, M, along with the maps $\phi_i = \pi \circ \phi_i^+$ form the inductive limit, where $\pi : \bigoplus_{i \in I} M_i \to M$ is the relevant quotient.

It is worth noting that we may think of this quotient as really creating an equivalence class of elements $m_i \in M_i$ and $m_j \in M_j$ whenever $\phi_{ki}(m_i) = \phi_{kj}(m_j)$ for some $k \geq i, j$. If we took this as our starting point, we could then have shown that a module structure could be placed on this set of equivalence classes, and then it would require proof that such a construction satisfied the universal property.

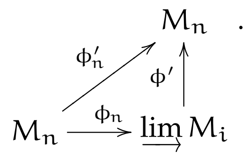
We're now in a position to state some basic properties of limits. The spirit under which all of these operate is to provide analogues to analytic limits.

Proposition 1.4. *If the directed set* I *has a largest element* n *(i.e.* $i \leq n$ *for all* $i \in I$*) then*

$$\phi_n : M_n \to \varinjlim M_i$$

is an isomorphism.

Proof. By the construction of $\varinjlim M_i$ we know that each element is the image of a ϕ_i, and $\phi_i = \phi_n \circ \phi_{ni}$. This shows that ϕ_n is surjective. However, we note that M_n equipped with the maps $\phi'_i = \phi_{ni}$ has the property that $\phi'_j \circ \phi_{ji} = \phi'_i$, so by universality there is some unique $\phi' : \varinjlim M_i \to M_n$ such that the following commutes:



However, as $\phi_n' = \phi_{nn}$, which is the identity, we see that the map ϕ_n must be injective. So ϕ_n is the desired isomorphism. □

Proposition 1.5 (Limits respect sums)**.** *Assume each* $M_i = A_i \oplus B_i$ *and if* $i \leq j$ *we can decompose* $\phi_{ji} = \alpha_{ji} + \beta_{ji}$*, where these are homomorphisms of the corresponding summands. If* $A = \varinjlim A_i$ *and* $B = \varinjlim B_i$*, which gives induced homomorphisms* $\alpha : A \to M$ *and* $\beta : B \to M$ *such that*

$$\alpha \circ \alpha_i = \phi_i|_A, \qquad \beta \circ \beta_i = \phi_i|B,$$

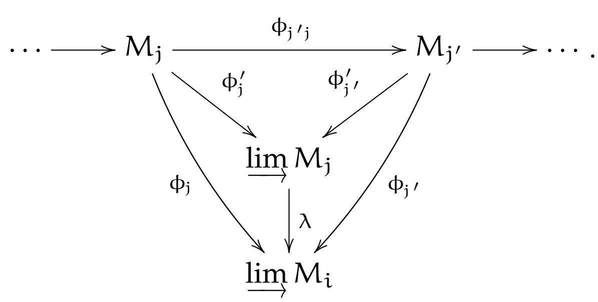
then $\alpha \oplus \beta : A \oplus B \to M$ *is an isomorphism.*

One easily constructs an inverse to this map by taking any any element of M and looking at it's inverse in some M_i. This decomposes into A_i and B_i components, and so one can send these to the corresponding element in $A \oplus B$. The only details involve checking that this process is independent of the preimage.

While the proof for the above is rather pedestrian, it is useful to know that limits respect sums, as they do in the analytic case. Another key property of direct limits will be an analog of the fact that limits of subsequences are equal to the full limit. To state this properly we need to introduce the concept of a *final* subset.

If I is our final set, we say that $J \subset I$ is a final subset if for each $i \in I$ there is a j such that $i \leq j$. Consider the example of compact subsets of $\mathbb{R}^n$ ordered by inclusion. Then the set of balls centered at the origin of radius $1, 2, \ldots$ forms a final subset, since any compact set will eventually be contained in a member of this subset.

Now, consider a direct system of modules $\{M_i\}_{i \in I}$ with maps $\phi_{ii'} : M_{i'} \to M_i, i, i' \in I, i' \leq i$. We can form the direct limit of such a system, $\varinjlim M_i$, with maps $\phi_i : M_i \to \varinjlim M_i$. A final subset $J \subset I$ induces another direct system of modules $\{M_j\}_{j \in J}$. As J is directed, it allows us to form a direct limit $\varinjlim M_j$ with maps $\phi_j' : M_j \to \varinjlim M_j$. As $\varinjlim M_i$ is a module with maps ϕ_j such that $\phi_j \circ \phi_{jj'}, j, j' \in J, j' \leq j$, the universal property allows us to find a unique homomorphism λ such that the following commutes:



In fact, λ is an isomorphism. To prove that λ is surjective is fairly easy, as the definition of J being final implies that any element of $\varinjlim M_i$ which is the image of some $m_i \in M_i$ under ϕ_i will be the image of some $\phi_{ji}(m) \in M_j, j \in J$ under ϕ'_j. Injectivity falls to the same type of argument, though requires a bit more checking. We'll record this as a lemma:

Proposition 1.6. *For any directed set M_i indexed by I, if $J \subset I$ is a final subset then the map*

$$\lambda : \varinjlim M_j \to \varinjlim M_i$$

is an isomorphism.

The last major fact we'll need about direct limits is that they commute with exact sequences.

Proposition 1.7. *Say A_i, B_i, and C_i are direct systems of modules indexed by I with maps $a_{ii'}, b_{ii'}$, and $c_{ii'}$ respectively. Further assume there exists a family of homomorphisms α_i and β_i such that the sequence*

$$A_i \xrightarrow{\alpha_i} B_i \xrightarrow{\beta_i} C_i$$

is exact and for all $i \leq i'$ the diagram below

$$\begin{array}{ccccc} A_i & \xrightarrow{\alpha_i} & B_i & \xrightarrow{\beta_i} & C_i \\ {\scriptstyle a_{i'i}}\downarrow & & {\scriptstyle b_{i'i}}\downarrow & & {\scriptstyle c_{i'i}}\downarrow \\ A_{i'} & \xrightarrow{\alpha_{i'}} & B_{i'} & \xrightarrow{\beta_{i'}} & C_{i'} \end{array}$$

commutes.

Then we obtain maps α and β and for all i we have the diagram below where the bottom row is exact and the diagram commutes for all i

$$\begin{array}{ccccc} A_i & \xrightarrow{\alpha_i} & B_i & \xrightarrow{\beta_i} & C_i \\ {\scriptstyle a_i}\downarrow & & {\scriptstyle b_i}\downarrow & & {\scriptstyle c_i}\downarrow \\ \varinjlim A_i & \xrightarrow{\alpha} & \varinjlim B_i & \xrightarrow{\beta} & \varinjlim C_i \end{array} .$$

We note that the bottom row is independent of i.

Proof. The proof amounts to a check. We begin by defining the maps $\psi_i = b_i \circ \alpha_i : A_i \to \varinjlim B_i$ and $\psi'_i = c_i \circ \beta_i : B_i \to \varinjlim C_i$. With this definition, we claim that $\psi_{i'} \circ a_{i'i} = \psi_i$ and $\psi'_{i'} \circ b_{i'i} = \psi'_i$. To verify the first such identity we choose any $x_i \in A_i$ and compute $\psi_{i'} \circ a_{i'i}(x_i) = b_{i'} \circ \alpha_{i'} \circ a_{i'i}(x_i)$ which by assumption (commutivity of the first diagram) is $b_{i'} \circ b_{i'i} \circ \alpha_i(x_i) = b_i \circ \alpha_i(x_i) = \psi_i(x_i)$. The second identity is similar. We then invoke the universal property on these ψ_i and ψ'_i to obtain the maps α and β, and by construction these maps make the second diagram commute.

Now let $x \in \varinjlim A_i$. We select some $x_i \in A_i$ such that $a_i(x_i) = x$. Then we observe that $\beta \circ \alpha(x) = \beta \circ \alpha \circ a_i(x_i)$. But as we've shown the second diagram commutes, this is just $c_i \circ \beta_i \circ \alpha_i(x_i)$, which is 0 by exactness. So $\mathrm{im}\alpha \subset \ker \beta$.

Next, assume $x \in \varinjlim B_i$ such that $\beta(x) = 0$. Let $x_i \in B_i$ such that $b_i(x_i) = x$. Then by commutativity of the second diagram, $0 = \beta \circ b_i(x_i) = c_i \circ \beta_i(x_i)$. We now need a technical sublemma which we shall not prove.

Lemma 1.8. *For any direct limit of modules M_i and maps $\phi_i : M_i \to \varinjlim M_i$, if $\phi_i(x) = 0$ then there exists some $i' \geq i$ such that $\phi_{i'i}(x) = 0$.*

One can prove this by teasing apart the construction of $\varinjlim M_i$.

In any case, we now choose such an i' as guaranteed by the lemma so that we have $0 = c_{i'i} \circ \beta_i(x_i) = \beta_{i'} \circ b_{i'i}(x_i)$, where the last equality is commutativity of the first diagram. But now we can invoke exactness at row i' to see that there exists some x_i' such that $\alpha_{i'}(x_i') = b_{i'i}(x_i)$. Hence we find that $\alpha \circ a_{i'}(x_i) = b_{i'} \circ \alpha_{i'}(x_i) = b_{i'} \circ b_{i'i}(x_i) = b_i(x_i) = x$, which finishes off the proof. □

An easy consequence of this lemma is the fact that exact sequences of any size (not just 3) pass to exact sequences in the limit. Using this fact, we can see that if all the α_i are injective, then so is α and similarly for surjectivity (hint: put 0s in the appropriate places).

Finally, we state a lemma which will be useful later on.

Lemma 1.9. *If $\phi_{i'i} : M_i \to M_{i'}$ is an isomorphism for all pairs $i \leq i'$ then the maps $\phi_i : M_i \to \varinjlim M_i$ are isomorphisms.*

Proof. We show surjectivity by contradiction. Assume ϕ_i is not surjective for some i. Then there is some $x \in \varinjlim M_i$ such that $x \neq \phi_i(m_i)$ for any $m_i \in M_i$. But by the construction of $\varinjlim M_i$, every element in $\varinjlim M_i$ is the image of some ϕ_j. Clearly, $j \not\leq i$, since we would have $\phi_j = \phi_i \circ \phi_{ij}$, which contradicts our assumption that x is not the image of some element under ϕ_i. But $j \not\geq i$, since we would then have $\phi_i = \phi_j \circ \phi_{ji}$, but since ϕ_{ji} is invertible we find that $\phi_j \circ \phi_{ji}^{-1} = \phi_j$, again contradicting our assumption. So ϕ_i must be surjective.

Now assume there exists some $m_i \in M_i$ such that $\phi_i(m_i) = 0$. But by lemma 1.8 there is some $j \geq i$ such that $\phi_{ji}(m_i) = 0$. But ϕ_{ji} is an isomorphism by our hypotheses, so $m_i = 0$. Thus ϕ_i is injective. □

1.2. **Cohomology with Compact Support.**

1.2.1. *Definition.* At this point we return to the world of topology, equipped to define our new type of cohomology. For a topological space X, we consider the directed set of subsets of X which are *compact* (we've already seen that such a set is directed), ordered by inclusion (we can thus use the notation $K \leq K'$ and $K \subset K'$ interchangeable for compact $K, K' \subset X$). For each compact K, we obtain an R-module $H^q(X, X - K)$. If $K \subset K'$ then the inclusion $(X, X-K') \to (X, X-K)$ induces a homomorpism $H^q(X, X-K) \to H^q(X, X-K')$ of modules. We recognize this to be a directed system, and so we may thus define

$$H_c^q(X) = \varinjlim H^q(X, X-K),$$

where we shall always understand that the directed set in question is compact $K \subset X$ ordered by inclusion. Such cohomology modules will be called *cohomology with compact support*.

Note here that the maps $\delta^* : H_c^*(X) \to H_c^{*+1}(X)$ are provided via the universal property. See the proof of the last proposition in the previous section for an idea of how this works. In fact, we can use this "pass to the limit" mantra of constructing maps via the universal property to bootstrap all our cohomology tools from the singular theory (quick quiz: why is the long exact sequence of a pair valid in cohomology?). However, to do this all rigourously, we need to make a few observations about the limitations of our new cohomology...

1.2.2. *Stickey Wickets.* What happens when we have a continuous map $f : X \to Y$ and want to induce a map in cohomology with compact support? Well, if this is going to work we're going to need $H^q(f)$ to make diagrams of the following form commute:

$$\begin{array}{ccc} H^q(X, X-K') & \longrightarrow & H_c^q(X) \\ \uparrow{\scriptstyle H^q(f)} & & \uparrow{\scriptstyle H_c^q(f)} \\ H^q(Y, Y-K) & \longrightarrow & H_c^q(Y) \end{array},$$

where hopefully $K' \subset X$ is some compact set, i.e. for every $K \subset Y$ there must be a corresponding $K' \subset X$ compact such that the above holds. Sadly, this is not always the case. For example, consider the map $f : (0,1) \to (0,1)$ which maps everything to $1/2$. If we choose $K = \{1/2\}$, then we note that f does not map any point into $(0,1) - \{1/2\}$. So, we must give up the use of arbitrary continuous f. However, we note that if f is *proper* in the sense that for any compact $K \subset Y$, $f^{-1}(K)$ is compact, then since f maps $X - f^{-1}(K)$ into $Y - K$ we're in good shape. A check verifies that the homomorphisms induced by f are compatible (in the universal property sense) with those induced by inclusion, so we are able to pass to the limit to define $H_c^q(f)$.

So, we know that if we limit ourselves to proper maps we won't have any trouble. Unfortunately, this excludes the inclusion map, which is deeply troubling. Luckily, for U open in X we can still define a map $H_c^q(U) \to H_c^q(X)$ (note the direction of the arrow). For any $K \subset U$ compact, we have the excision $H^q(X, X-K) \to H^q(U, U-K)$. Using the inverse of this isomorphism, we get the collection of diagrams

$$\begin{array}{ccc} H^q(X, X-K) & \longrightarrow & H^q_c(X) \\ \uparrow & & \uparrow \\ H^q(U, U-K) & \longrightarrow & H^q_c(U) \end{array},$$

and since the isomorphism commutes with the maps induced by inclusion (another check), we can pass to the limit to get a unique map $H^q_c(U) \to H^q_c(X)$ which makes the above commute.

1.2.3. *The Computation.* So with this rather unwieldy algebraic structure, can we actually compute anything? Indeed, we can, but in general direct computation is fairly challenging. In any case, we have

Proposition 1.10. *For* $i \neq n$, $H^i_c(\mathbb{R}^n) \cong 0$, *and further* $H^n_c(\mathbb{R}^n) \cong R$.

Remembering that we constructed cohomology with compact support specifically for the duality theorem to hold, we are thus delighted that the above result holds, since this is the duality result we expect on $\mathbb{R}^n$. We give a proof as follows:

Proof. We note that computing a quantity like $H^i_c(\mathbb{R}^n, \mathbb{R}^n - K)$ for arbitrary compact K is a hopeless task. To circumvent this problem, we shall use the notion of a final subset. Indeed, we claim that the set $\{B(k)\}$ of closed balls centered at the origin and of radius k constitutes a final subset of the compact subsets of $\mathbb{R}^n$, ordered by inclusion. This claim is really Heine-Borel in disguise, noting that every compact $K \subset B(k)$ for some k sufficiently large. Thus, by proposition (1.6) we see that instead of taking a direct limit indexed by all compact $K \subset \mathbb{R}^n$, we can instead take the limit indexed by $B(n)$, and the resulting module will be isomorphic to $H^i_c(\mathbb{R}^n)$.

We compute $H^i(\mathbb{R}^n, \mathbb{R}^n - B(k))$. First, we note that $H_i(\mathbb{R}^n, \mathbb{R}^n - B(k)) \cong \tilde{H}_i(\mathbb{R}^n - B(k)) \cong \tilde{H}_i(S^n)$ be the sequence of a pair, say. Then, an appeal to the U.C.T. gives that $H^i(\mathbb{R}^n, \mathbb{R}^n - B(k)) \cong 0$ if $i \neq n$ and R otherwise. Finally, the maps $H^i(\mathbb{R}^n, \mathbb{R}^n - B(k)) \to H^i(\mathbb{R}^n, \mathbb{R}^n - B(k+1))$ are isomorphisms for all i, which one can see by any number of standard arguments (naturality of the sequence of a pair, say), but more simply by noting that the generator of $H^n(\mathbb{R}^n, \mathbb{R}^n - B(k))$ is sent to $H^n(\mathbb{R}^n, \mathbb{R}^n - B(k+1))$. So, we can conclude that $H^i_c(\mathbb{R}^n)$ is as claimed, invoking lemma (1.9). □

It is worth noting that this result shows cohomology with compact support is *not* a homotopy invariant, since $\mathbb{R}^n$ is contractible. We may blame this on the fact that restricting to proper maps removes certain homotopies we would have in the regular theory.

1.3. **Cap Products and the Duality Map.** Morally speaking, we want the map for Poincaré duality to be given by $\zeta_m \cap : H^i(M) \to H_{n-i}(M)$, and for compact manifolds, it is. As we've mentioned, this is not going to hold for non-compact manifolds. So, we do the next best thing and "pass to the limit". To do this, however, we'll need to understand how the cup product interacts with direct limits.

First, some notation. For the sequel M is an n-manifold with an R-orientation (specifically, we've chosen one). We view this orientation as a section of the R-orientation sheaf, and thus may restrict this section to K and use the isomorphism $H_n(M, M-K) \cong \Gamma K$ (22.24 in Greenberg) to obtain $\zeta_K \in H_n(M, M-K)$.

Observe that the cup product is a map $\cap : H_{p+q}(X, A) \times H^p(X, A) \to H_q(X)$ (24.26 in Greenberg), so this allows us to construct the map $\zeta_K \cap$ for any compact $K \subset M$ by

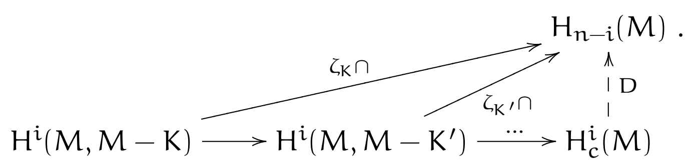
$$\zeta_K \cap : H^i(M, M-K) \to H_{n-i}(M)$$

where

$$c \mapsto \zeta_K \cap c.$$

The most important property of this map is that for all $K \subset K' \subset M$, the solid part of the diagram below commutes.

(1)



Assuming this is true, we may then pass to the limit to obtain a map $D : H^i_c(M) \to M_{n-i}(M)$, and this will be the Poincaré duality map.

So how do we see that the diagram commutes? To begin with, we'll need to recall that the cap product satisfies a somewhat awkward naturality property, namely for $f : (X, A) \to (Y, B)$, $a \in H_{p+q}(X, A)$ and $b \in H^p(Y, B)$, we have

$$H_q(f)[a \cap H^p(f)b] = H_{p+q}(f)(a) \cap b. \tag{2}$$

You'll recall that we've proved part of this result (in the non-relative case) on the final exam. In any case, see Greenberg 24.24 or Hatcher section 3.3 for further discussion.

Thus, if we consider f to be the identity map on M and let $a = \zeta_{K'}$, then (2) becomes

$$H_{n-i}(id)[\zeta_{K'} \cap H^i(id)c] = H_n(id)(\zeta_{K'}) \cap c,$$

where $c \in H^i(M, M-K)$. We then note that $H_{n-i}(id) : H_{n-i}(M) \to H_{n-i}(M)$ is the identity map in homology, so the above reduces further to become

$$\zeta_{K'} \cap H^i(id)c = H_n(id)(\zeta_{K'}) \cap c.$$

Thus, if we can show that $H_n(id)(\zeta_{K'}) = \zeta_K$ then we'll have shown the desired commutativity, as c was arbitrary.

To show this, we need to return to the definition of ζ_K and $\zeta_{K'}$. Viewing an R-orientation as a section of the R-orientation sheaf, the restriction of this section to an element of ΓK followed by the inverse of the canonical isomorphism $j_K : H_n(M, M-K) \to \Gamma K$ defines

ζ_K (similarly for $\zeta_{K'}$). Understanding this, the fact that the following diagram commutes (see Greenberg page 166) give us our result

$$\begin{array}{ccc} H_n(M, M-K') & \xrightarrow{j_{K'}} & \Gamma K' \\ {\scriptstyle H_n(id)}\downarrow & & \downarrow{\scriptstyle r} \\ H_n(M, M-K) & \xrightarrow{j_K} & \Gamma K \end{array},$$

where here r is the restriction map. To see this, note that by definition $\zeta_{K'}$ maps to the restriction of the R-orientation (section) on K'(an element of $\Gamma K'$), which then restricts to the restriction of the R-orientation to K (an element of ΓK), which then maps to ζ_K under j_K^{-1}, by definition. As the square commutes, this element achieved going clockwise around the square agrees with $H_n(id)(\zeta_{K'})$. Thus, we've shown that

$$\zeta_{K'} \cap H^i(id)(c) = \zeta_K \cap c,$$

as desired.

1.3.1. *A Side Note.* We've gone to all the trouble to define the duality map D as the direct limit of modules. However, in the case where the manifold is actually compact, $H^i_c(M) = H^i(M)$. Thus, we would hope that D is actually the map ζ_M in this case. This is true, and may be seen by noting that if M is compact then M is a final element among the directed set of compact subsets, ordered by inclusion. Thus, the direct limit of the $H^i(M, M-K)$ modules is isomorphic to the direct limit taken over the single module $H^i(M, M-M) \cong H^i(M)$ by (1.6), and a check of the universal property verifies, in a manner very similar to the computation done above, that this isomorphism is in fact equality.

1.3.2. *Mayer-Vietoris and the Duality Map.* The key technical tool in proving Poincaré duality is the use of a Mayer Vietoris argument. Specifically, given open sets U and V and their intersection $B = U \cap V$, if Poincaré duality holds for these sets then it holds for the union $Y = U \cup V$.

To realize this, we'll need to work in the setting of singular homology and cohomology before we can pass to cohomololgy with compact support. I'll keep the notation in this section in line with Greenberg 218 to 220 to facilitate reading that proof. With the given U, V, and B, let K and L be compact in U and V, respectively. Using the Mayer-Vietoris sequence of the triple $(Y, Y-K, Y-L)$ we obtain the diagram

(3)

$$\begin{array}{ccccccccc} \longleftarrow H^{q+1}(B, B-K\cap L) & \longleftarrow & H^q(Y, Y-K\cup L) & \longleftarrow & H^q(U, U-K)\oplus H^q(V, V-L) & \longleftarrow & H^q(B, B-K\cap L) & \longleftarrow & . \\ \downarrow{\scriptstyle \zeta_{K\cap L}} & & \downarrow{\scriptstyle \zeta_{K\cup L}} & & \downarrow{\scriptstyle \zeta_K\oplus\zeta_L} & & \downarrow{\scriptstyle \zeta_{K\cap L}} & & \\ \longleftarrow H_{n-q-1}(B) & \longleftarrow & H_{n-q}(Y) & \longleftarrow & H_{n-q}(U)\oplus H_{n-q}(V) & \longleftarrow & H_{n-q}(B) & \longleftarrow & \end{array}$$

The observant reader, however, will note that the top sequence doesn't appear to be the Meyer-Vietoris sequence in cohomology. In particular, the cohomology MV sequence for the triple $(Y, Y-K, Y-L)$ should look like

$$\longleftarrow H^{q+1}(Y, Y-K\cap L) \longleftarrow H^q(Y, Y-K\cup L) \longleftarrow H^q(Y, Y-K)\oplus H^q(Y, Y-L) \longleftarrow H^q(Y, Y-K\cap L) \longleftarrow \ .$$

Luckily, we can pass from this sequence to the prior one by using a series of excisions. For example, $(U, U-K) = (Y-(Y-U), Y-K-(Y-U)) \to (Y, Y-K)$ is an excision, as is $(B, B-K\cap L) = (Y-(Y-U\cap V), Y-K\cap L-(Y-U\cap V)) \to (Y, Y-K\cap L)$.

Next, we need to verify the commutativity of (3). We focus only on the two rightmost squares, as the leftmost is an ungodly mess. To begin with, we note that the (nonboundary) maps of the upper sequence in (3) are all induced from the relevant inclusions, the reason being that that the excision isomorphisms all come from inclusions as well, so tracing the defining maps shows this result. Thus, we are left to show a result of the form

$$H_{n-q}(i_1)(\zeta_{K\cap L}\cap a)\oplus H_{n-q}(i_2)(\zeta_{K\cap L}\cap a) = \zeta_K\cap H^q(i_1)(a)\oplus \zeta_L\cap H^q(i_2)(a),$$

where here $a\in H^q(B, B-K\cap L)$ and i_1 and i_2 are the relevant injections for each factor of the direct sum. To prove this result, one uses the naturality property for cap products (2) as well as the definition of ζ_K, ζ_L, and $\zeta_{K\cap L}$. The spirit and technique are the same as that used to define D, but the details are sufficiently soul-breaking that I won't include them here. The same goes for commutativity of the middle square.

Finally, we wish to pass to the limit to obtain a diagram using compactly supported cohomology and the D map. To do this, we simply observe that every compact set in Y can be written as the union $K\cup L$ of compact sets in U and V, respectively (this is a point-set fact). Thus, we get the induced diagram

(4)
$$\begin{array}{ccccccccc}
\longleftarrow & H_c^{q+1}(B) & \longleftarrow & H_c^q(Y) & \longleftarrow & H_c^q(U)\oplus H_c^q(V) & \longleftarrow & H_c^q(B) & \longleftarrow \ . \\
 & \downarrow D & & \downarrow D & & \downarrow D\oplus D & & \downarrow D & \\
\longleftarrow & H_{n-q-1}(B) & \longleftarrow & H_{n-q}(Y) & \longleftarrow & H_{n-q}(U)\oplus H_{n-q}(V) & \longleftarrow & H_{n-q}(B) & \longleftarrow
\end{array}$$

The rows of this diagram are exact by propositions 1.5 and 1.7, (finally all the abstract nonsense becomes useful). The commutativity of the diagram follows from commutativity of diagrams of the form (3) (well, to be honest, we can really only claim sign commutativity of all diagrams, as the leftmost square sign commutes. But as we didn't prove this and it won't affect our result, we don't care.) Thus, we note that by assumption all maps but $D: H_c^q(Y)\to H_{n-q}(Y)$ are isomorphisms. But by the five lemma, this map is as well.