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Topology Review Notes 20

1. Cohomology Ring of the Torus

We let ϕ be the map that identifies the edges in the diagram, $\phi : I^2 \to T$. We will show that $z = \phi(A_0A_1B_1) - \phi(A_0B_0B_1)$ is a cycle which generates $H_2(T)$ and hence $\zeta = [z]$. First we show that z is a cycle. $\delta z = \phi(A_1B_1) - \phi(A_0B_1) + \phi(A_0A_1) - \phi(B_0B_1) + \phi(A_0B_1) - \phi(A_0B_0) = 0$.

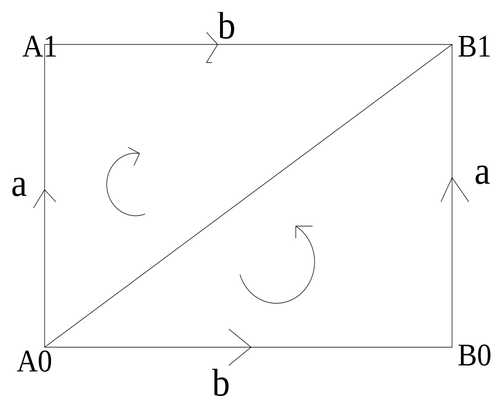

FIGURE 1. Identified edges in the torus construction.

From our original calculation of $H_2(T)$ we have the isomorphisms $H_2(T) \to H_2(T, S^1 \bigvee S^1) \to H_2(E^2, S^1) \to H_1(S^1)$. Thus we have $\{z\}$ maps to $(A_0A_1) + (A_1B_1) - (B_0B_1) - (A_0B_0)$ which generates $H_1(S^1)$. Hence $\{z\}$ is a generator of $H_2(T)$.

Assuming R is a PID, $H^1(T) = R \oplus R$ (since $H^1(T) = Hom(H_1(T; \mathbb{Z}, R))$. The generators of $H_1(T)$ are α and β where α is the homology class of the loop $\phi(A_0A_1)$ and β is the homology class of the loop $\phi(A_0B_0)$. Since any element of $H^1(T)$ is determined by it's values at α and β, the generators of $H^1(T)$ are α^* and β^* where $[\alpha, \alpha^*] = 1$, $[\beta, \alpha^*] = 0$, $[\beta, \beta^*] = 1$, and $[\alpha, \beta^*] = 0$.

We will show that $\alpha^* \cup \beta^*$ generates $H^2(T)$. $H^2(T)$ is isomorphic to R by sending any element of $H^2(T)$ to the value it takes on z, the generator of $H_2(T)$. Therefore, we need to check that $[z, \alpha^* \cup \beta^*] = 1$.

$[z, \alpha^* \cup \beta^*] = [\phi(A_0A_1B_1), \alpha^* \cup \beta^*] - [\phi(A_0B_0B_1), \alpha^* \cup \beta^*] = [\phi(A_0A_1), \alpha^*][\phi(A_1B_1), \beta^*] - [\phi(A_0B_0), \alpha^*][\phi(B_0B_1)$ from the definition of cup product. Since $\phi(A_1B_1)$ is the same loop as $\phi(A_0B_0)$ and $\phi(B_0B_1)$ is the same loop as $\phi(A_0A_1)$ we have $[\alpha, \alpha^*][\beta, \beta^*] - [\beta, \alpha^*][\alpha, \beta^*] = 1 + 0 = 1$. Therefore $[z, \alpha^* \cup \beta^*] = 1$ and $\alpha^* \cup \beta^*$ generates $H^2(T)$. Thus we know that α^* and β^* generate $H^\bullet(T)$, so it only remains to find the relations on α^* and β^* to determine the cup product structure.

$[z, \alpha^* \cup \alpha^*] = [\alpha, \alpha^*][\beta, \alpha^*] - [\beta, \alpha^*][\alpha, \alpha^*] = 0$
$[z, \beta^* \cup \beta^*] = [\alpha, \beta^*][\beta, \beta^*] - [\beta, \beta^*][\alpha, \beta^*] = 0$
$\alpha^* \cup \beta^* = -\beta^* \cup \alpha^*$ since $\alpha^* \cup \beta^* = (-1)^{pq} \beta^* \cup \alpha^*$

2. Cohomology Ring of the Double Torus

Let D denote the double torus.

$$H^q(D) = H_q(D) = \begin{cases} 0 & \text{if } q > 2, \\ R & \text{if } q = 0, 2 \\ R \oplus R \oplus R \oplus R & \text{if } q = 1. \end{cases} \tag{1}$$

From our original calculation of $H_2(D)$ we have the isomorphisms $H_2(D) \to H_2(D, S^1 \bigvee S^1 \bigvee S^1 \bigvee S^1) \to H_2(E^2, S^1) \to H_1(S^1)$. Thus it suffices to exhibit a relative 2-cycle on (E^2, S^1) whose boundary represents a generator of $H_1(S^1)$. Define ϕ :octagon$\to$ D to be the quotient mapping that identifies the edges of our diagram.

Let $\alpha, \beta, \gamma, \delta$ be the generators of $H_1(D)$ and let $\alpha^*, \beta^*, \gamma^*, \delta^*$ be their duals, and hence the generators of $H^1(D)$.

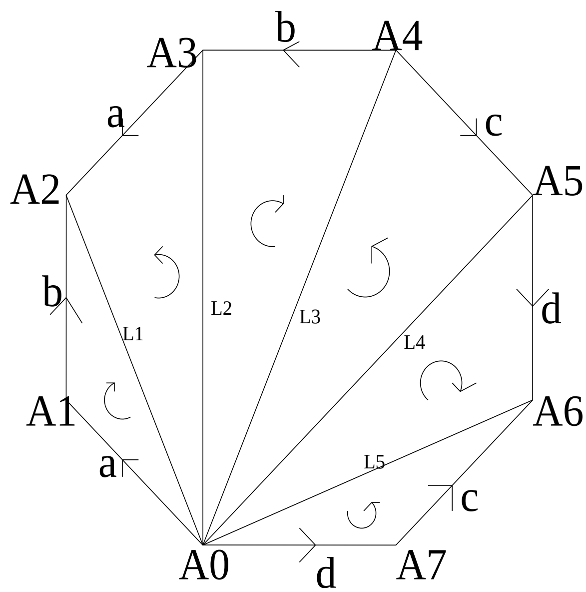

FIGURE 2. Identified edges in the double torus construction.

Let $z = \phi(A_0A_1A_2) - \phi(A_0A_3A_2) + \phi(A_0A_3A_4) - \phi(A_0A_5A_4) + \phi(A_0A_5A_6) - \phi(A_0A_7A_6)$. We will show that this choice of z is a 2-cycle on D whose homology class ζ generates $H_2(D)$. $\partial z = \phi(A_0A_1) + \phi(A_1A_2) - \phi(A_3A_2) + \phi(A_3A_4) - \phi(A_5A_4) + \phi(A_5A_6) - \phi(A_7A_6) - \phi(A_0A_7) = 0$. Therefore z is a cycle. We can show that ζ generates $H_2(D)$ by showing that the under the isomorphisms above, z maps to a generator of $H_1(S^1)$. z maps to $A_0A_1 + A_1A_2 - A_3A_2 + A_3A_4 - A_5A_4 + A_5A_6 - A_7A_6 - A_0A_7$ which generates $H_1(S^1)$. Then $\zeta = \{z\}$ where ζ is the fundamental class of D. We will show that $\alpha^* \cup \beta^*$ generates $H^2(D)$ and thus that $\alpha^*, \beta^*, \gamma^*, \delta^*$ generates the cohomology ring of D.

$[\zeta, \alpha^* \cup \beta^*] = [\alpha, \alpha^*][\beta, \beta^*] - [L_2, \alpha^*][\alpha, \beta^*] + [L_2, \alpha^*][-\beta, \beta^*] - [L_4, \alpha^*][\gamma, \beta^*] + [L_4, \alpha^*][\delta, \beta^*] - [\delta, \alpha^*][\gamma, \beta^*] = 1 - [L_2, \alpha^*]$.

Note that $\alpha^* \cup \alpha^* = (-1)^{|\alpha|}\alpha^* \cup \alpha^*$ so $\alpha^* \cup \alpha^* = 0$. Thus, $0 = [\zeta, \alpha^* \cup \alpha^*] = [\alpha, \alpha^*][\beta, \alpha^*] - [L_2, \alpha^*][\alpha, \alpha^*] + [L_2, \alpha^*][-\beta, \alpha^*] - [L_4, \alpha^*][\gamma, \alpha^*] + [L_4, \alpha^*][\delta, \alpha^*] - [\delta, \alpha^*][\gamma, \alpha^*] = -[L_2, \alpha^*]$ since the L_2 term is the only one to survive. So $[L_2, \alpha^*] = 0$. Combining with the above we have $[\zeta, \alpha^* \cup \beta^*] = 1 - [L_2, \alpha^*] = 1$. Thus $\alpha^* \cup \beta^*$ generates $H^2(D)$. So we have the ring generated by $\alpha^*, \beta^*, \gamma^*, \delta^*$ where $\alpha^* \cup \beta^*$ generates $H^2(D)$.

3. Cohomology Ring of the $\mathbb{CP}^n$

We will show that $H^*(\mathbb{CP}^n) = \mathbb{Z}[\gamma]/\gamma^{n+1}$, where γ generates $H^2(\mathbb{CP}^n)$, i.e. γ generates the cohomology ring of $\mathbb{CP}^n$, and is truncated at height $2n+1$. We will prove this by induction on n. Let γ generate $H^2(\mathbb{CP}^n)$.

Base case: $n = 2$. (In case $n = 1$, $\mathbb{CP}^1 = S^2$, and the result is clear.) Recall that $H^q(\mathbb{CP}^n)$ is 0 if q is odd, so $q = 1$ is 0. We need only to show that $\gamma \cup \gamma$ generates $H^4(\mathbb{CP}^n)$. Let $\zeta \in H_4(\mathbb{CP}^n)$ be the fundamental class. Then $\zeta \cap \gamma$ generates H_2 since this is the isomorphism of Poincaré duality. Thus we have $[\zeta \cap \gamma, \gamma]$ generates R. But $[\zeta \cap \gamma, \gamma] = [\zeta, \gamma \cup \gamma]$ so $[\zeta, \gamma \cup \gamma]$ also generates R. Therefore $\gamma \cup \gamma$ generates H^4.

Now we use the fact that inclusion $\mathbb{CP}^{n-1} \to \mathbb{CP}^n$ induces isomorphisms in all dimensions less than $2n-2$. Dimension $2n-1$ is odd and thus 0, and dimension $2n$ follows from above. γ^{n-1} generates H^{2n-2} so $\zeta \cap \gamma^{n-1}$ generates H_2. So $[\zeta \cap \gamma^{n-1}, \gamma] = [\zeta, \gamma^{n-1} \cup \gamma]$ generates R so γ^n generates H^{2n}.

4. Cohomology Ring of the $\mathbb{RP}^n$

Let $R = \mathbb{Z}_2$. We will show $H^*(\mathbb{RP}^n, \mathbb{Z}_2) = \mathbb{Z}_2[\gamma]/\gamma^{n+1}$. Then

$$H^q(\mathbb{RP}^n) = H_q(\mathbb{RP}^n) = \begin{cases} 0 & \text{if } q > n, \\ \mathbb{Z}_2 & \text{if } q =\leq n. \end{cases} \tag{2}$$

Let γ generate $H^1(\mathbb{RP}^n)$. Then γ generates the cohomology ring of $\mathbb{RP}^n$. We will prove this by induction on n.

Base case: $n = 2$. (In case $n = 1$, $\mathbb{RP} = S^1$, and the result is clear.) We need only to show that $\gamma \cup \gamma$ generates $H^2(\mathbb{RP}^n)$. Let $\zeta \in H_2(\mathbb{RP}^2)$ be the fundamental class. Then $\zeta \cap \gamma$ generates H_1 since this is the isomorphism of Poincaré duality. Thus we have $[\zeta \cap \gamma, \gamma]$ generates $\mathbb{Z}_2$. But $[\zeta \cap \gamma, \gamma] = [\zeta, \gamma \cup \gamma]$ so $[\zeta, \gamma \cup \gamma]$ also generates $\mathbb{Z}_2$. Therefore $\gamma \cup \gamma$ generates H^2.

Now we use the fact that inclusion $\mathbb{RP}^{n-1} \to \mathbb{RP}^n$ induces isomorphisms in all dimensions less than $n-1$. Dimensions n and $n-1$ follows from above. γ^{n-2} generates H^{n-2} so $\zeta \cap \gamma^{n-2}$ generates H_2. So $[\zeta \cap \gamma^{n-2}, \gamma \cup \gamma] = [\zeta, \gamma^n]$ generates $\mathbb{Z}_2$ so γ^n generates H^n. We can use a similar argument for the $n-1$ case. $[\zeta, \gamma^n] = [\zeta \cap \gamma^{n-1}, \gamma]$ so $\zeta \cap \gamma^{n-1}$ generates H_1 so γ^{n-1} generates H^{n-1}.

5. Cohomology Ring of the $\mathbb{HP}^n$

Let γ generate $H^4(\mathbb{HP}^n)$. We will show that $H^*(\mathbb{HP}^n) = \mathbb{Z}[\gamma]/\gamma^{n+1}$, i.e. γ generates the cohomology ring of $\mathbb{HP}^n$. We will prove this by induction on n. Recall that $H^q(\mathbb{HP}^n)$ is 0 if q is not divisible by 4, and R otherwise.

Base case: $n = 2$. (In case $n = 1$, $\mathbb{HP} = S^4$, and the result is clear.) We need only to show that $\gamma \cup \gamma$ generates $H^4(\mathbb{CP}^n)$. Let $\zeta \in H_8(\mathbb{HP}^2)$ be the fundamental class. Then $\zeta \cap \gamma$ generates H_4 since this is the isomorphism of Poincaré duality. Thus we have $[\zeta \cap \gamma, \gamma]$ generates R. But $[\zeta \cap \gamma, \gamma] = [\zeta, \gamma \cup \gamma]$ so $[\zeta, \gamma \cup \gamma]$ also generates R. Therefore $\gamma \cup \gamma$ generates H^8.

Now we use the fact that inclusion $\mathbb{HP}^{n-1} \to \mathbb{HP}^n$ induces isomorphisms in all dimensions less than $4n-1$. Dimension $4n-1$ is odd and thus 0, and dimension $4n$ follows from above. γ^{n-1} generates H^{4n-4} so $\zeta \cap \gamma^{n-1}$ generates H_4. So $[\zeta \cap \gamma^{n-1}, \gamma] = [\zeta, \gamma^{n-1} \cup \gamma]$ generates R so γ^n generates H^{4n}.

6. $\mathbb{Z}_k$ ORIENTABILITY IMPLIES $\mathbb{Z}$ ORIENTABILITY

Assume X is $\mathbb{Z}_k$ orientable. We have a collection of $\{(U_i, \overline{\alpha_i})\}$ where U_i is homeomorphic to $\mathbb{R}^n$ and $\overline{\alpha_i}$ generates $H_n(X, X - U_i; \mathbb{Z}_k)$. If $x \in U_i \bigcap U_j$ then $j_x^{U_i}(\overline{\alpha_i}) = j_x^{U_j}(\overline{\alpha_j})$. Suppose that X is not $\mathbb{Z}$ orientable. Then for any choice of generators α_i' of $H_n(X, X - U_i; \mathbb{Z})$, there is an $x \in U_i \bigcap U_j$ such that $j_x^{U_i}(\alpha_i') \neq j_x^{U_j}(\alpha_j')$. Since, by definition, a unit in a ring has an inverse, there exists $m \in \mathbb{Z}_k$ such that $m\overline{\alpha_i} = \overline{1}$. It is clear that $\{(U_i, \overline{m\alpha_i})\}$ also defines an orientation. Therefore, without loss of generality we may assume that $\overline{\alpha_i} = \overline{1}$, since $U_i \cong \mathbb{R}^n$.

Let θ be the map from $H(X, X - U; \mathbb{Z})$ to $H(X, X - U; \mathbb{Z}_k)$ sending $\sum a_i\sigma_i$ to $\sum \overline{a_i}\sigma_i$, where $\overline{a_i}$ is the class of a_i under projection into $\mathbb{Z}_k$. Note that $H_n(X, X - U; \mathbb{Z}) = \mathbb{Z}$ and $H_n(X, X - U; \mathbb{Z}_k) = \mathbb{Z}_k$ and θ is reduction mod k.

$$\begin{array}{ccc} S_n(X, X - U_i; \mathbb{Z}) & \xrightarrow{\rho} & S_n(X, X - U_i; \mathbb{Z}_k) \\ \downarrow i & & \downarrow i \\ S_n(X, X - x; \mathbb{Z}) & \xrightarrow{\rho} & S_n(X, X - x; \mathbb{Z}_k) \end{array}.$$

We show that this diagram commutes, where ρ denotes reduction mod k, and i is the inclusion map (note that ρ is a chain map as we've shown in a previous exercise). In the first direction $i(\rho(\Sigma\alpha_i\sigma_i + S_n(X - U_i))) = i(\Sigma\overline{\alpha_i}\sigma_i + S_n(X - U_i)) = \Sigma\overline{\alpha_i}\sigma_i + S_n(X - x)$. And in the other direction $\rho(i(\Sigma\alpha_i\sigma_i + S_n(X - U_i))) = \rho(\Sigma\alpha_i\sigma_i + S_n(X - x)) = \Sigma\overline{\alpha_i}\sigma_i + S_n(X - x)$. Thus, we can pass to homology , and the following diagram commutes.

$$\begin{array}{ccc} H_n(X, X - U_i; \mathbb{Z}) & \xrightarrow{\theta} & H_n(X, X - U_i; \mathbb{Z}_k) \\ \downarrow j_x^{U_i} & & \downarrow j_x^{U_i} \\ H_n(X, X - x; \mathbb{Z}) & \xrightarrow{\theta} & H_n(X, X - x; \mathbb{Z}_k) \\ \uparrow j_x^{U_j} & & \uparrow j_x^{U_j} \\ H_n(X, X - U_j; \mathbb{Z}) & \xrightarrow{\theta} & H_n(X, X - U_j; \mathbb{Z}_k) \end{array}.$$

θ maps generators of $\mathbb{Z}$, $(1, -1)$ to generators of $\mathbb{Z}_k$ $(\overline{1}, \overline{k-1})$ and since k is not 2, these are two distinct generators. Therefore we let $\alpha_i' = 1$ so that $\theta(\alpha_i') = \overline{\alpha_i}$ and we also choose $\alpha_j' \in \{1, -1\}$ so that $\theta(\alpha_j') = \overline{\alpha_j}$. By assumption, $j_x^{U_i}(\alpha_i') \neq j_x^{U_j}(\alpha_j')$ and both are units, so $\theta(j_x^{U_i}(\alpha_i')) \neq \theta(j_x^{U_j}(\alpha_j'))$ since θ is bijective from the set $\{1, -1\}$ to the set $\{\overline{1}, \overline{k-1}\}$. By commutativity of the above diagram, $\theta(j_x^{U_i}(\alpha_i')) = j_x^{U_i}(\theta(\alpha_i'))$ and $\theta(j_x^{U_j}(\alpha_j')) = j_x^{U_j}(\theta(\alpha_j'))$. We also have that $\theta(\alpha_i') = \overline{\alpha_i'}$ and $j_x^{U_i}(\overline{\alpha_i'}) = j_x^{U_j}(\overline{\alpha_j'})$. Thus $\theta(j_x^{U_i}(\alpha_i')) = j_x^{U_i}(\overline{\alpha_i'}) = j_x^{U_j}(\overline{\alpha_j'}) = \theta(j_x^{U_j}(\alpha_j'))$. Contradiction.

7. EULER CHARACTERISTICS OF COMPACT ORIENTABLE MANIFOLDS

Recall that the Euler characteristic is given by: $\chi(X) = \sum_q (-1)^q \beta_q$. Then from Exercise 23.40 we know that with $\mathbb{Z}$ coefficients, $H^q \cong F_q \oplus T_{q-1}$ where F_q is the quotient module of H_q by its torsion submodule, and T_{q-1} is the torsion submodule of H_{q-1}. Then using Poincare Duality:

$$\beta_q = \mathrm{rank}(H_q; \mathbb{Z}) = \mathrm{rank}(H^{n-q}; \mathbb{Z}) = \mathrm{rank}(F_{n-q}; \mathbb{Z}) = \mathrm{rank}(H_{n-q}; \mathbb{Z}) = \beta_{n-q}.$$

We have proved

Corollary 1 (26.8)**.** *If X is a compact orientable n-manifold, then the Betti numbers of X satisfy* $\beta_q = \beta_{n-q}$ *for all* q*.*

The fact that many of these β_q's match off in pairs will enable us to partially classify $\chi(X)$ for these spaces.

Corollary 2 (26.10)**.** *If X is an odd-dimensional compact orientable n-manifold, then* $\chi(X) = 0$*.*

Let $n = 2j+1$. We just expand $\chi(X)$ and then use Corollary 26.8:

$$\chi(X) = \sum_{k=0}^{2j+1}(-1)^k\beta_k = \sum_{k=0}^{j}(-1)^k\beta_k + \sum_{k=j+1}^{2j+1}(-1)^k\beta_k = \sum_{k=0}^{j}(-1)^k\beta_k + \sum_{k=0}^{j}(-1)^{n-k}\beta_{n-k}$$
$$= \sum_{k=0}^{j}(-1)^k\beta_{n-k} + (-1)^{n-k}\beta_{n-k}$$

Then we notice that when k is odd, $n-k$ is even, and when k is even, $n-k$ is odd, therefore the two β_{n-k} terms always have opposite sign, meaning $\chi(X) = 0$.

Corollary 3 (26.11)**.** *If X is an even-dimension compact orientable n-manifold, and the dimension is not divisible by 4, then $\chi(X)$ is even.*

Let $n = 4k+2$. Using the same reasoning as Cor 2 (26.10), then all but one of the Betti numbers double up, because when t is odd, $4k+2-t$ is also odd. The only term left out of this doubling up is the $2k+1^{\text{th}}$, meaning all we need to show is that β_{2k+1} is even. When working with coefficients in a field, homology and cohomology are dual vector spaces. Also, since β measures rank, it will be convenient to choose coefficients where there can be no torsion, i.e. where rank is equivalent to dimension. Therefore we choose $\mathbb{Q}$ coefficients, and then

$$\beta_{2k+1} = \mathrm{rank}(H_{2k+1}(X,\mathbb{Q})) = \dim(H^{2k+1}(X,\mathbb{Q}))$$

Let's assume X is connected, in which case $H_0(X,\mathbb{Q}) = H^n(X,\mathbb{Q}) = \mathbb{Q}$. Prof. Lin proved in class that the cup product is a non-degenerate skew-symmetric bilinear form on $H^{2k+1}(X;\mathbb{Q})$.

Lemma 1 (26.11a)**.** *If we are given a skew symmetric bilinear nondegenerate form $<\,,\,>$ from $V \mathrm{x} V$ to $\mathbb{Q}$, then $\dim(V)$ is even.*

Choose a basis $x_1, x_2, ...x_n$ for V. Let $a_{ij} =< x_i, x_j >$, and let $A = (a_{ij})$, the matrix of the a_{ij}'s. $a_{ij} = -a_{ji}$ since our form is skew-symmetric, so $A^T = -A$. To prove the lemma, we need another lemma.

Lemma 2 (26.11b)**.** *In the case of cohomology and cup product, $\det(A) \neq 0$.*

Let $b_i =$ the i^{th} row of A. If $\sum_i c_i b_i = 0$, then $\sum_i c_i a_{ij} = 0 \;\forall\; 1 \leq j \leq n$. Then $\sum_i c_i < x_i, x_j >= 0 \;\forall\; j$, but since the form is nondegenerate, this implies that $\sum_i c_i x_i = 0$. But the x_i are linearly independent, so $c_i = 0 \;\forall\; i$. This means that the rows of A are linearly independent, which is equivalent to $\det(A) \neq 0$.

Now, $\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A)$. Dividing both sides by $\det(A)$, we get $1 = (-1)^n$. So n must be even.

Then $\dim(H^{2k+1})$ is even, meaning β^{2k+1} is even, so $\chi(X)$ is even. Quite a bit of work for one corollary!

We have now talked about Euler characteristic for compact orientable manifolds of dimension 1, 2, or 3 mod 4. Do ones with dimension 4k also have even (or perhaps odd) characteristic? Consider $\mathbb{CP}^{2k}$. $\chi(\mathbb{CP}^{2k}) = \beta_{2k} = \mathrm{rank}(H_{2k}(\mathbb{CP}^{2k},\mathbb{Z}) = \mathrm{rank}(\mathbb{Z}) = 1$ which is odd. But $\chi(S^4,\mathbb{Z}) = 2$, which is even. Thus no conclusion can be made about the odd or evenness of manifolds with dimension a multiple of 4. These formulas also do not hold if we remove our orientability assumption. Consider $\mathbb{P}^2$. If Corollary 26.11 held for non-orientable spaces, $\chi(\mathbb{P}^2)$ would be even, but it equals 1!

Theorem 1 (26.18)**.** *Let M be an n-dimensional, compact connected manifold. Prove the following:*

- ♣ *If M is orientable, then $H_{n-1}(M;\mathbb{Z})$ is torsion-free.*
- ♠ *If M is not orientable, then $H_n(M;\mathbb{Z}_k) = 0$ for k odd.*
- ♡ *If M is not orientable, then the torsion subgroup of $H_{n-1}(M;\mathbb{Z}))$ is $\mathbb{Z}_2$. (**NOTE:** This is different than the statement in the book – it asks for the torsion of $H_{n-1}(M;\mathbb{Z}_2)$, but this is meaningless.)*

$\diamondsuit$ *If* M *is not orientable, then* $H_1(M;\mathbb{Z}_2)$ *is nonzero.*

If M is orientable, then

$$H_{n-1}(M;\mathbb{Z}) \cong H^1(M;\mathbb{Z}) \cong \mathrm{Hom}(H_1(M;\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Ext}(H_0(M;\mathbb{Z}),\mathbb{Z}),$$

where the first step is by Poincaré Duality. Since $\mathrm{Hom}(R,\mathbb{Z})$ is torsion-free for rings R and $\mathrm{Ext}(\mathbb{Z}^r,\mathbb{Z}) = 0$ for any r, $H_{n-1}(M;\mathbb{Z})$ has no torsion. (♣)

Now suppose M is not orientable. As we showed before, if a manifold is orientable if and only if it is $\mathbb{Z}_3$-orientable. Exercise 22.39 extends this result to $\mathbb{Z}_k$-orientability for *any* $k \geq 3$, and it isn't hard to extend Evan's proof for $k = 3$ to any ring with at least two units. So if any of $H_n(M;\mathbb{Z}_k)$ are nonzero for $k \geq 3$, this would contradict the supposition that $H_n(M;\mathbb{Z}) = 0$. So we have $H_n(M;\mathbb{Z}_k) = 0$ for all $k \geq 3$. (♠)

For the rest of the problem, M is taken to be non-orientable, and we will use the result to frequently make the substitutions $H_n(M;\mathbb{Z}) = 0$, $H_n(M;\mathbb{Z}_2) = \mathbb{Z}_2$ (all compact connected manifolds are $\mathbb{Z}_2$-orientable), and $H_n(M;\mathbb{Z}_k) = 0$ for $k \geq 3$.

For computing the torsion subgroup of $H_{n-1}(M;\mathbb{Z})$, we will be using the two Bocksteins for homology seen before – namely, those for the sequences $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_k \to 0$ and $0 \to \mathbb{Z}_k \to \mathbb{Z}_{k^2} \to \mathbb{Z}_k \to 0$. We will be studying these chains for the cases $k = 2$ and $k \geq 2$, but first we make a quick observation.

Since M is $\mathbb{Z}_2$-orientable (regardless of $\mathbb{Z}$-orientability), we have $H_1(M;\mathbb{Z}_2) \cong H^{n-1}(M;\mathbb{Z}_2)$ by Poincaré Duality. Also, since $\mathbb{Z}_2$ is a field, $H^{n-1}(M;\mathbb{Z}_2)$ and $H_{n-1}(M;\mathbb{Z}_2)$ are dual vector spaces. Connecting, we have $H_1(M;\mathbb{Z}_2) \cong H_{n-1}(M;\mathbb{Z}_2)$, so to finish off $\diamondsuit$ we need only establish $H_{n-1}(M;\mathbb{Z}_2) \neq 0$.

We have the short exact ladder:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times k} & \mathbb{Z} & \xrightarrow{\pi} & \mathbb{Z}_k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \mathbb{Z}_k & \xrightarrow{\times k} & \mathbb{Z}_{k^2} & \xrightarrow{\pi} & \mathbb{Z}_k & \longrightarrow & 0
\end{array}$$

Passing to homology we get a long exact ladder:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H_n(M;\mathbb{Z}) & \xrightarrow{\pi} & H_n(M;\mathbb{Z}_k) & \xrightarrow{\beta_1} & H_{n-1}(M;\mathbb{Z}) & \xrightarrow{\times k} & H_{n-1}(M;\mathbb{Z}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H_n(M;\mathbb{Z}_{k^2}) & \xrightarrow{\pi} & H_n(M;\mathbb{Z}_k) & \xrightarrow{\beta_2} & H_{n-1}(M;\mathbb{Z}_k) & \xrightarrow{\times k} & H_{n-1}(M;\mathbb{Z}_{k^2}) & \longrightarrow & \cdots
\end{array}$$

where β_1 and β_2 are the Bockstein maps.

In the case where $k = 2$, M is orientable, so the bottom row above becomes $0 \to \mathbb{Z}_2 \to H_{n-1}(M;\mathbb{Z}_2) \to H_{n-1}(M;\mathbb{Z}_4)$, which by exactness implies $H_{n-1}(M;\mathbb{Z}_2) \neq 0$. ($\diamondsuit$)

Right, now if $k \geq 3$, the portion of the top row above is

$$0 \longrightarrow 0 \xrightarrow{\beta_1} H_{n-1}(M;\mathbb{Z}) \xrightarrow{\times k} H_{n-1}(M;\mathbb{Z})$$

Since $0 = \mathrm{im}\ \beta_1 = \ker\ (\times\ k)$, $H_{n-1}(M;\mathbb{Z})$ has no $\mathbb{Z}_k$-torsion for any $k \geq 3$. If we look at the same row for $k = 2$, we get

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{\beta_1} H_{n-1}(M;\mathbb{Z}) \xrightarrow{\times 2} H_{n-1}(M;\mathbb{Z})$$

Since im $\partial = \ker \beta_1$, $H_{n-1}(M;\mathbb{Z})$ contains $\mathbb{Z}_2$-torsion. Therefore the torsion subgroup of $H_{n-1}(M;\mathbb{Z})$ is some number of copies of $\mathbb{Z}_2$. But since $\mathbb{Z}_2 = \text{im } \beta_1 = \ker(\times 2)$, we must have exactly one copy of $\mathbb{Z}_2$ in $H_{n-1}(M;\mathbb{Z})$. (♡)

Theorem 2 (26.19)**.** *If M is an orientable 3-manifold with* $H_1(M;\mathbb{Z}) = 0$*, then M has the homology of a 3-sphere.*

Because M is orientable, $H_3(M;\mathbb{Z}) = \mathbb{Z}$; we assume M is connected, so also $H_0(M;\mathbb{Z}) = \mathbb{Z}$. Then all that is left to show is that $H_2(M;\mathbb{Z}) = 0$.

By Poincare Duality, $H^2(M;\mathbb{Z}) = H_1(M;\mathbb{Z})$ which is zero by assumption.

Then using the Universal Coefficient Theorem, we see:

$$0 = H^2(M;\mathbb{Z}) = \mathrm{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Ext}(H_1(M;\mathbb{Z}),\mathbb{Z})$$

which means that $\mathrm{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z}) = 0$, i.e. $H_2(M;\mathbb{Z})$ has no free part. Again using Poincare Duality, then UCT:

$$H^3(M;\mathbb{Z}) = H_0(M;\mathbb{Z}) = \mathbb{Z}$$

$$\mathbb{Z} = H^3(M;\mathbb{Z}) = \mathrm{Hom}(H_3(m;\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Ext}(H_2(M;\mathbb{Z}),\mathbb{Z}).$$

$$= \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \oplus \mathrm{Ext}(H_2(M;\mathbb{Z}),\mathbb{Z}) = \mathbb{Z} \oplus \mathrm{Ext}(H_2(M;\mathbb{Z}),\mathbb{Z})$$

Therefore $\mathrm{Ext}(H_2(M;\mathbb{Z}),\mathbb{Z}) = 0$, i.e. $H_2(M;\mathbb{Z})$ has no torsion. But if it has no free part, and no torsion, then it has nothing at all. So $H_2(M;\mathbb{Z}) = 0$ and M has the homology of a 3-sphere.

Theorem 3 (26.21)**.** *Let* M *be a orientable compact connected manifold. The cup product pairing* $H^p(M) \oplus H^q(M) \longrightarrow H^n(M)$, $p + q = n$ *and coefficients in a field, has the property that for a fixed* $x \in H^p(M)$, $x \cup y = 0$ *for all* $y \in H^q(M)$ *only if* $x = 0$.

Proof: As M is compact and connected by Poincare duality we know there exists a generator $\zeta \in H^p(M)$ such that $\zeta\cap : H^q(M) \longrightarrow H_p(M)$ is an isomorphism. By assumption we have for all y

$$0 = [\zeta, x \cup y] = [\zeta \cap x, y] \tag{3}$$

Suppose that $x \neq 0$. Now since our coefficients are in a field $H^q(M) \cong \mathrm{Hom}_F(H_q(M), F)$. Moreover $x \neq 0$ and $\zeta\cap$ an isomorphism implies that $\zeta \cap x \neq 0$ as well. Therefore there exists some $y \in H^q(M)$ such that $[\zeta \cap x, y] \neq 0$. (This last assertion follows from basic linear algebra: given some nonzero element v in a vector space V_F there is some linear functional $f : V \longrightarrow F$ such that $f(v) \neq 0$.) This contradicts (1). Therefore $x = 0$.■

Theorem 4 (26.22)**.** $\mathbb{CP}^{2n}$ *admits no orientation reversing homotopy equivalence.*

We know that $H^*(\mathbb{CP}^{2n}) = \mathbb{Z}[\gamma]/\gamma^{2n+1}$, $\gamma \in H^2(\mathbb{CP}^{2n})$ by 26.12 (proved earlier in this packet of notes).

Let ζ be the fundamental class of $\mathbb{CP}^{2n}$, so $< \zeta >= H_{4n}(\mathbb{CP}^{2n},\mathbb{Z})$.

Proof by contradiction: Assume there exists f such that $H_{4n}(f)(\zeta) = -\zeta$. Let ζ^* be the dual of ζ, $< \zeta^* >= H^{4n}(\mathbb{CP}^{2n})$. Since γ generates all of $H^*(\mathbb{CP}^{2n})$, it must be that either $\gamma^{2n} = \zeta^*$ or $-\zeta^*$. Note that since $[\zeta, \zeta^*] = 1$,

$$[H_{4n}(f)(\zeta), \zeta^*] = [-\zeta, \zeta^*] = -1 = [\zeta, H^{4n}(f)(\zeta^*)]$$

then $H^{4n}(f)(\zeta^*) = -\zeta^*$ and $H^{4n}(f)(-\zeta^*) = \zeta^*$, so in either case, $H^{4n}(f)(\gamma^{2n}) = -\gamma^{2n}$. (*)

$H^2(f)$ is an isomorphism, so it must send a generator to a generator, i.e. $H^2(f)(\gamma) = \gamma$ or $-\gamma$.

Case 1: $H^2(f)(\gamma) = \gamma$. Then $H^{4n}(f)(\gamma^{2n}) = (H^{4n}(f)(\gamma))^{2n} = (\gamma)^{2n} = \gamma^{2n}$ since cohomology is a ring homomorphism.

Case 2: $H^2(f)(\gamma) = -\gamma$. $H^{4n}(f)(\gamma^{2n}) = (H^{4n}(f)(\gamma))^{2n} = (-\gamma)^{2n} = \gamma^{2n}$.

In either case, $H^{4n}(f)(\gamma^{2n}) = \gamma^{2n}$. But (*) says it equals $-\gamma^{2n}$! Contradiction. Therefore no orientation reversing homotopy equivalence exists.

Theorem 5 (26.23). *We have maps $S^3 \to S^2$ and $S^7 \to S^4$ with Hopf invariant one. Prove that for these maps*

$$\begin{array}{ccc} S^{2n-1} & \stackrel{f}{\to} & S^n \\ -1\downarrow & & \downarrow -1 \\ S^{2n-1} & \stackrel{f}{\to} & S^n \end{array}$$

cannot commute even up to homotopy, where -1 means a map of degree -1.

Proof Let us assume that we have maps of degree -1, s and t, such that this commutes for $n = 2$ (repsectively $n = 4$). Then by 21.20 there exists a $\gamma : \mathbb{C}P^2 \to \mathbb{C}P^2 (= Cf)$ (respectively $\gamma : \mathbb{H}P^2 \to \mathbb{H}P^2$) such that the following diagram commutes

$$\begin{array}{ccccccccc} \leftarrow & H^q(S^3) & \leftarrow & H^q(S^2) & \leftarrow & H^{q-1}(\mathbb{C}P^2) & \leftarrow & H^q(S^3) & \leftarrow \\ & \uparrow s^* & & \uparrow t^* & & \uparrow \gamma^* & & \uparrow s^* & \\ \leftarrow & H^q(S^3) & \leftarrow & H^q(S^2) & \leftarrow & H^{q=1}(\mathbb{C}P^2) & \leftarrow & H^q(S^3) & \leftarrow \end{array}$$

(repsectively with S^7, S^4 and $\mathbb{H}P^2$). We recall that the cohomology ring of $\mathbb{C}P^n$ (respectively $\mathbb{H}P^n$) is $\mathbb{Z}[\beta]/\beta^3$ where β is a generator of level 2 (respecitvely β is a generator of level 4) and $\beta \cup \beta = \zeta$, a generator of $H^4(\mathbb{C}P^2)$ (respectively $H^8(\mathbb{H}P^2)$). As all squares in the diagram commute we have that

$$\begin{array}{ccc} H^2(S^2) & \stackrel{e^*}{\longleftarrow} & H^2(\mathbb{C}P^2) \\ \uparrow t^* & & \uparrow \gamma^* \\ H^2(S^2) & \stackrel{e^*}{\longleftarrow} & H^2(\mathbb{C}P^2) \end{array}$$

commutes (respectively in degree 4 with S^4 and $\mathbb{H}P^2$) and we note that $t^* : H^2(S^2) \to H^2(S^2)$ is multiplication by -1. By taking $\eta \in H^2(S^2)$ and looking at how it evaluates on $x \in H_2(S^2)$ we have that $t^*(\eta)(x) = \eta(t_*x) = \eta(-x) = -\eta(x)$ as t has degree -1, and in fact note that if $s : S^n \to S^n$ has degree -1 then $s^* : H^n(S^n) \to H^n(S^n)$ is multiplication by -1 (so this works in the $n = 4$ case as well). Thus if $f^*(\beta) = \eta$, then $t^*e^*(\beta) = -\eta$ and hence $f^*\gamma^*(\beta) = -\eta$ so as e^* is an isomorphism $\gamma^*(\beta) = -\beta$. Likewise we have that

$$\begin{array}{ccc} H^4(\mathbb{C}P^2) & \leftarrow g^* & H^3(S^3) \\ \uparrow \gamma^* & & \uparrow s^* \\ H^4(\mathbb{C}P^2) & \leftarrow g^* & H^3(S^3) \end{array}$$

commutes (respectively with degrees 7 and 8, and $\mathbb{H}P^2, S^7$.) As above s^* is multiplication by -1. We consider $\zeta = \beta \cup \beta$. We have that (as g^* is an isomorphism) that $\zeta = g^*(\vartheta)$ so $\gamma^*(\zeta) = g^*s^*(\vartheta) = g^*(-\vartheta) = -\zeta$. Now, if such maps existed, then γ^* would be a ring map. But this would imply that

$$-\zeta = \gamma^*(\zeta) = \gamma^*(\beta \cup \beta) = \gamma^*(\beta) \cup \gamma^*(\beta) = (-\beta) \cup (-\beta) = \beta \cup \beta = \zeta$$

which is a contradiction, so no such s and t of degree -1 can exist such that this commutes.

Theorem 6 (26.24). *A necessary condition for the lens space $L(p, q)$ to admit an orientation reversing homotopy equivalence is that -1 be a quadratic residue mod p, i.e. there is an integer λ such that $-1 \equiv \lambda^2 \pmod p$. (We consider only the case when p is prime.)*

Proof. Let $L := L(p, q)$ be the lens space. We record here the homology and cohomology of L over several rings.

m	$H_m(L;\mathbb{Z})$	$H_m(L;\mathbb{Z}_p)$	$H^m(L;\mathbb{Z})$	$H^m(L;\mathbb{Z}_p)$	$H^m(L;\mathbb{Z}_{p^2})$
3	$\mathbb{Z}$	$\mathbb{Z}_p$	$\mathbb{Z}$	$\mathbb{Z}_p$	$\mathbb{Z}_{p^2}$
2	0	$\mathbb{Z}_p$	$\mathbb{Z}_p$	$\mathbb{Z}_p$	$\mathbb{Z}_p$
1	$\mathbb{Z}_p$	$\mathbb{Z}_p$	0	$\mathbb{Z}_p$	$\mathbb{Z}_p$
0	$\mathbb{Z}$	$\mathbb{Z}_p$	$\mathbb{Z}$	$\mathbb{Z}_p$	$\mathbb{Z}_{p^2}$

Of course all higher-degree homology and cohomology modules are 0, since L is a 3-dimensional manifold.

Suppose now that $f : L \to L$ is an orientation reversing homotopy equivalence. This means $H_3(f;\mathbb{Z}) = -1$, i.e. the isomorphism induced by f on $H_3(L;\mathbb{Z})$ is multiplication by -1. Thus also $\mathrm{Hom}(H_3(f;\mathbb{Z}), \mathbb{Z}_p) = -1$. Now $\mathrm{Ext}(H_2(L;\mathbb{Z}), \mathbb{Z}_p) = \mathrm{Ext}(0, \mathbb{Z}_p) = 0$, so the naturality in the universal coefficient theorem gives us

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^3(L;\mathbb{Z}_p) & \xrightarrow[h]{\cong} & \mathrm{Hom}(H_3(L;\mathbb{Z}), \mathbb{Z}_p) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle \mathrm{Hom}(H_3(f;\mathbb{Z}),\mathbb{Z}_p)=-1} & & \\ 0 & \longrightarrow & H^3(L;\mathbb{Z}_p) & \xrightarrow[h]{\cong} & \mathrm{Hom}(H_3(L;\mathbb{Z}), \mathbb{Z}_p) & \longrightarrow & 0 \end{array}$$

where the map h is an isomorphism. Thus $h(f^*(1)) = -h(1) = h(-1)$, so $f^*(1) = -1$. So $f^* : H^3(L;\mathbb{Z}_p) \to H^3(L;\mathbb{Z}_p)$ is multiplication by -1.

Now consider the short exact sequence

$$0 \longrightarrow \mathbb{Z}_p \xrightarrow{\times p} \mathbb{Z}_{p^2} \xrightarrow{\text{mod } p} \mathbb{Z}_p \longrightarrow 0.$$

This induces the long exact sequence

$$\begin{array}{ccccccccc} 0 \longleftarrow & H^3(L;\mathbb{Z}_p) & \xleftarrow{\eta} & H^3(L;\mathbb{Z}_{p^2}) & \xleftarrow{\theta} & H^3(L;\mathbb{Z}_p) & \xleftarrow{\beta} & H^2(L;\mathbb{Z}_p) & \longleftarrow \cdots \\ & = \mathbb{Z}_p & & = \mathbb{Z}_{p^2} & & = \mathbb{Z}_p & & = \mathbb{Z}_p & \end{array}$$

where β is the Bockstein homomorphism. Now η is onto, hence its kernel is a subgroup of $\mathbb{Z}_{p^2}$ with order p, which must be isomorphic to $\mathbb{Z}_p$. So im $\theta = \mathbb{Z}_p$, and thus θ must be injective. So $\beta = 0$. Continuing, we get

$$\begin{array}{ccccccccc} 0 \longleftarrow & H^2(L;\mathbb{Z}_p) & \xleftarrow{\eta} & H^2(L;\mathbb{Z}_{p^2}) & \xleftarrow{\theta} & H^2(L;\mathbb{Z}_p) & \xleftarrow{\beta} & H^1(L;\mathbb{Z}_p) & \longleftarrow \cdots \\ & = \mathbb{Z}_p & & = \mathbb{Z}_p & & = \mathbb{Z}_p & & = \mathbb{Z}_p & \end{array}$$

(we have reused the symbols η, θ, β). Now η is onto, hence it must be an isomorphism; thus $\theta = 0$, and β is onto. Hence $\beta : H^1(L;\mathbb{Z}_p) \xrightarrow{\cong} H^2(L;\mathbb{Z}_p)$ is an isomorphism.

Now let x be a generator of $H^1(L;\mathbb{Z}_p) \cong \mathbb{Z}_p$. Then βx is a generator of $H^2(L;\mathbb{Z}_p)$.

Claim 1. $x \cup \beta x$ *is a generator of* $H^3(L;\mathbb{Z}_p)$.

Proof of claim. (This is the only place where we use our assumption that p was prime.) Suppose $x \cup \beta x$ is not a generator of $H^3(L;\mathbb{Z}_p) \cong \mathbb{Z}_p$; then it must be 0. Now any element $y \in H^1(L;\mathbb{Z}_p)$ can be written as $y = cx$ for some integer c. Then we have $y \cup \beta x = cx \cup \beta x = c(x \cup \beta x) = 0$ for all $y \in H^1(L;\mathbb{Z}_p)$. But $\mathbb{Z}_p$ is a field, so by Exercise 26.21, this can only happen if $\beta x = 0$. We have $x \neq 0$ and β an isomorphism, so this is not the case. □

Now, since $f^* = -1$, we have $f^*(x \cup \beta x) = -(x \cup \beta x)$. But f^* is a ring homomorphism, so

$$\begin{aligned} f^*(x \cup \beta x) &= f^*(x) \cup f^*(\beta x) \\ &= f^*(x) \cup \beta(f^*(x)) \end{aligned}$$

since β is a natural homomorphism and thus commutes with f^*. Since x generates $H^1(L;\mathbb{Z}_p)$, $f^*(x) = \lambda x$ for some integer λ. Thus

$$\begin{aligned} f^*(x \cup \beta x) &= \lambda x \cup \beta(\lambda x) \\ &= \lambda x \cup \lambda \beta x \\ &= \lambda^2 (x \cup \beta x). \end{aligned}$$

So we have $-(x \cup \beta x) = \lambda^2 (x \cup \beta x)$. But $x \cup \beta x$ is a generator of $\mathbb{Z}_p$, hence invertible, so we must have $\lambda^2 \equiv 1 \pmod{p}$. □

Note 1. *If* p *is not prime, the problem is apparently still true. We only need to prove Claim 1, that* $x \cup \beta x$ *generates* $H^3(L, \mathbb{Z}_p)$. *This in turn relies on the following unproved assertion made in Hatcher, whose proof is not immediately apparent to us.*

Claim 2. *The natural map*

$$\begin{aligned} H^1(L;\mathbb{Z}_p) &\to \mathrm{Hom}(H_1(L;\mathbb{Z}_p),\mathbb{Z}_p) \\ y &\mapsto [\cdot, y] \end{aligned}$$

is an isomorphism.

You cannot use the universal coefficient theorem to prove this, since $\mathbb{Z}_p$ may not be a PID (it has zero divisors if p is not prime).

Now since $H_1(L;\mathbb{Z}_p) \cong \mathbb{Z}_p$, we know $\mathrm{Hom}(H_1(L;\mathbb{Z}_p),\mathbb{Z}_p)$ contains some isomorphisms. In fact, if x is a generator of $H^1(L;\mathbb{Z}_p)$, then $[\cdot, x]$ is a generator of $\mathrm{Hom}(H_1(L;\mathbb{Z}_p),\mathbb{Z}_p)$ and so must be an isomorphism itself (if the generator is not an isomorphism, no other element can be one).

We can prove Claim 1 in a way much like the proof of Exercise 26.21. We look at $\beta x \cup x = -x \cup \beta x$ for convenience.

Proof of Claim 1. Suppose $\beta x \cup x$ is not a generator of $H^3(L;\mathbb{Z}_p) \cong \mathbb{Z}_p$. Then there is an integer $0 < r < p$ with $r(\beta x \cup x) = 0$. In particular, if ζ is a generator of $H_3(L;\mathbb{Z}_p)$ (i.e. an orientation class), then we have $0 = [\zeta, r\beta x \cup x] = [\zeta \cap r\beta x, x]$ since $\cup$ and $\cap$ are adjoint. Since $[\cdot, x]$ is an isomorphism, $\zeta \cap r\beta x = 0$. But $\zeta \cap \cdot$ is an isomorphism by Poincaré duality, so $r\beta x = 0$. βx is a generator of $H^2(L;\mathbb{Z}_p) \cong \mathbb{Z}_p$, so this is impossible. □

Theorem 7 (26.25). *If* $f : S^n \to S^n$ *commutes with the antipodal map then* f *has odd degree.*

Proof. Case 1: n is odd. For the sake of contradiction, assume f has even degree. Then f induces a map $g : \mathbb{P}^n \to \mathbb{P}^n$ such that the diagram commutes.

$$\begin{array}{ccc} S^n & \xrightarrow{f} & S^n \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ \mathbb{P}^n & \xrightarrow{g} & \mathbb{P}^n \end{array}.$$

Since $-f(x) = f(-x)$ by assumption, $g([x]) = [f(x)]$ is well-defined. Then

$$\begin{array}{ccc} H_n(S^n;\mathbb{Z}) & \xrightarrow{f_*} & H_n(S^n;\mathbb{Z}) \\ \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} \\ H_n(\mathbb{P}^n;\mathbb{Z}) & \xrightarrow{g_*} & H_n(\mathbb{P}^n;\mathbb{Z}) \end{array}.$$

Since n is odd $H_n(\mathbb{P}^n;\mathbb{Z}) = \mathbb{Z}$. Assuming f has degree $2l$, we take a generator 1 of $H_n(S^n;\mathbb{Z})$. $p_* f_*(1) = 4l$ and $p_*(1) = 2$ so $g_*(2) = 4l$. Therefore $g_*(1) = 2l$.

Now we look at cohomology with $\mathbb{Z}_2$ coefficients.

$H^n(\mathbb{P}^n;\mathbb{Z}) \leftarrow H^n(\mathbb{P}^n;\mathbb{Z})$

$\mathrm{Hom}(H_n(\mathbb{P}^n;\mathbb{Z});\mathbb{Z}_2) \leftarrow \mathrm{Hom}(H_n(\mathbb{P}^n;\mathbb{Z})\mathbb{Z}_2)$

$\mathrm{Hom}(g_*, \mathbb{Z}_2) = 0$ since g_* is multiplication by 2. Let γ be the height $n+1$ generator of the cohomology ring. If $g^*(\gamma) = \gamma$ then $g^*(\gamma^n) = \gamma^n$ since g^* is a ring map. But $g^*(\gamma^n) = 0$ by the above. Therefore $g^*(\gamma) = 0$. Using the Universal Coefficient Theorem, $\mathrm{Ext}(\mathbb{Z}, \mathbb{Z}_2) = 0$ and the fact that $H_1(\mathbb{P}^n, \mathbb{Z}) = \pi_1(\mathbb{P}^n)$ we have:

$$\begin{array}{ccc} H^1(\mathbb{P}^n;\mathbb{Z}_2) & \xrightarrow{\cong} & \mathrm{Hom}(\pi_1(\mathbb{P}^n;\mathbb{Z});\mathbb{Z}_2) \\ \downarrow g^*=0 & & \downarrow \mathrm{Hom}(g_\#,\mathbb{Z}_2) \\ H^1(\mathbb{P}^n;\mathbb{Z}_2) & \xrightarrow{\cong} & \mathrm{Hom}(\pi_1(\mathbb{P}^n);\mathbb{Z}_2) \end{array} .$$

This implies that $\mathrm{Hom}(g_\#,\mathbb{Z}_2)=0$ and thus we have $g_\#\pi_1(\mathbb{P}^n)=0$ so there exists a lift g'. So $pg'=g$. Therefore we have two lifts of gp. $pg'p=gp$ and $pf=gp$. and then $pg'p=pf$. So $g'p(x)=\pm f(x)$. If $g'p(x)=f(x)$ then both lifts agree at x so the two lifts must be the same. If $g'p(x)=-f(x)$ then $g'p(-x)=g'p(x)=-f(x)=f(-x)$ and hence they agree at $-x$. Therefore the lifts are the same by the unique lifting theorem. Now, since $-f(x)=f(-x)$, $-g'p(x)=g'p(-x)=g'p(x)$ which is a contradiction. Therefore f must have odd degree.

Case 2: n is even.

$f:S^n\to S^n$

Then $\Sigma f:\Sigma S^n\to\Sigma S^n$. Since $\Sigma S^n=S^{n+1}$, $\Sigma f:S^{n+1}\to S^{n+1}$. $\Sigma S^n=\frac{S^n\times[-1,1]}{S^n\times\{-1\}\&S^n\times\{1\}}=S^{n+1}$.

$\Sigma f(x,t)=(f(x),t)$, $a(x,t)=(ax,-t)$. $a\Sigma f(x,t)=a(f(x),t)=(af(x),-t)=(fa(x),-t)=\Sigma f(ax,-t)=(\Sigma f)a(x,t)$ So $a\Sigma f=(\Sigma f)a$. Additionally, if deg f is j, we have

$$\begin{array}{ccc} H_n(S^n) & \xrightarrow{f_*} & H_n(S^n) \\ \downarrow\cong & & \downarrow\cong \\ H_{n+1}(\Sigma S^n) & \xrightarrow{\Sigma f_*} & H_{n+1}(\Sigma S^n) \end{array} .$$

which shows that Σf has the same degree as f.

Therefore by case 1, Σf has odd degree, and thus f has odd degree as well.

□

Theorem 8 (26.26)**.** *An orientable, compact, connected n-manifold M is* spherical *if there exists $f:S^n\to M$ such that $H(f)\zeta_S=k\zeta_M$ for some $k\neq 0$. Prove that if $p\nmid k$, then f induces an isomorphism of mod p homology. Conclude that $H_i(M;\mathbb{Z})$ is finite for $1\le i<n$.*

Proof. We assume p prime (else the theorem isn't true...). We'll first need to compute $H(f):H_n(S^n;\mathbb{Z}_p)\to H_n(M;\mathbb{Z}_p)$. As usual, when attempting to relate $\mathbb{Z}$ homology and $\mathbb{Z}_p$ homology, we use the long exact sequence obtained from the short exact sequence:

(4) $$0\longrightarrow\mathbb{Z}\xrightarrow{\times p}\mathbb{Z}\xrightarrow{\pi}\mathbb{Z}_p\longrightarrow 0 ,$$

where π is the relevant projection.

From this sequence and its functorality, we obtain the commuting square:

$$\begin{array}{ccc} H_n(S^n;\mathbb{Z}) & \xrightarrow{\pi_*} & H_n(S^n;\mathbb{Z}_p) \\ \downarrow H(f) & & \downarrow H(f) \\ H_n(M;\mathbb{Z}) & \xrightarrow{\pi_*} & H_n(M;\mathbb{Z}_p) \end{array} .$$

Thus, we learn that $H(f)(\pi_*(\zeta_S))=k\pi_*(\zeta_M)$, where here k indicates the equivalence class of $k \mod p$. In particular, we note that the projection map must send generators to generators, so $\pi_*(\zeta_S)$ and $\pi_*(\zeta_M)$ both generate their respective groups. As $p\nmid k$, we see that k must be in $\mathbb{Z}_p^\times$. Thus, k is a unit, and hence $k\pi_*(\zeta_M)$ must also generate its group. Thus, we may conclude that f induces isomorphism for the top homology $\mod p$.

For all homology lower than n, we note that $H_q(S^n;\mathbb{Z}_p)$ is trivial. Thus, if we show that $H_q(f;\mathbb{Z}_p)$ is surjective, we'll have shown isomorphism. To do this, let $a\in H_q(M;\mathbb{Z}_p)$. By Poincaré duality, there exists some $b \in H^{n-q}(M;\mathbb{Z}_p)$ such that $a = \pi_*(\zeta_M) \cap b$. Hence,

$$\begin{aligned} a &= \pi_*(\zeta_M) \cap b \\ &= k^{-1}k\pi_*(\zeta_M) \cap b && \text{(using } k \text{ a unit)} \\ &= k^{-1}H_n(f)(\pi_*(\zeta_S)) \cap b && \text{(above computation)} \\ &= k^{-1}H_q(f)[\pi_*(\zeta_S) \cap H^{n-q}(f)(b)] && \text{(24.24 Greenberg).} \end{aligned}$$

So we get surjectivity, and hence f induces an isomorphism in $\mod p$ homology.

To conclude that $H_i(M;\mathbb{Z})$ is finite for $1 \le i < n$, we'll again need the long exact sequence induced from (4). Hence, for any q,

$$\cdots \longrightarrow H_{q+1}(M;\mathbb{Z}_p) \xrightarrow{\beta} H_q(M;\mathbb{Z}) \xrightarrow{\theta_*} H_q(M;\mathbb{Z}) \xrightarrow{\pi_*} H_q(M;\mathbb{Z}_p) \longrightarrow \cdots,$$

where θ is the map induced by $\times p$. For $q < n-1$, $H_{q+1}(M;\mathbb{Z}_p)$ and $H_q(M;\mathbb{Z}_p)$ are both 0, so θ_* is an isomorphism. In fact, this holds for $q = n-1$ as well by noting that β in this case is the zero map (can be seen via exactness of the sequence coupled with surjectivity of π_* in the n-th homologies).

Assume, for contradiction, that H_i has infinite order. Now, by the theorem of finitely generated abelian groups,

$$H_i(M;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \text{Torsion}.$$

But as θ_* maps this group to itself by multiplication by p, we observe that such a map cannot be surjective (observe that the generator of the first $\mathbb{Z}$ factor is not in the image of θ_*). As we showed that θ_* is an isomorphism, we reach a contradiction. Thus, $H_i(M;\mathbb{Z})$ must be finite.

We can also deduce the allowable torsion arguing as above. For a group $\mathbb{Z}_q$ mapped to itself by $\times p$, we see that this is an isomorphism only if $p \mod q$ is a unit of $\mathbb{Z}_q$. This condition is equivalent to p, q being relatively prime. Of course, if we let p vary over all primes which fail to divide k, we find that the only allowable torsion elements are those which are relatively prime to all such p. Thus, the allowable torsion factors must each be of an order which is the product only of primes which divide k. □

As a final note, we observe that if we don't assume p to be prime, then the theorem fails. To see this, choose $p = 4$, $k = 2$, and choose $M = S^1$. Then we have $H_1(f)(\pi_*(\zeta_{S^1})) = 2\pi_*(\zeta_{S^1})$ in $\mathbb{Z}_4$ homology. This clearly cannot be an isomorphism.

Topology 26.20

Chris Tiee, Evan Füller

Problem 0.1. *Let M be a compact, connected, and nonorientable 3-manifold. Then $H_1(M,\mathbb{Z})$ is infinite.*

Solution. To keep everything compact (ha ha) $H_q(M)$ always will mean homology with $\mathbb{Z}$ coefficients. By (26.17) the homology modules of M are all finitely generated since M is compact. Therefore we shall bust up $H_k(M)$, by the Fundamental Theorem of Finitely Generated Abelian Groups (FTAG), into its free and torsion constituents (the invariant factors) separating out the even and odd torsion parts:

$$H_k(M) \cong \mathbb{Z}^{\beta_k} \oplus \bigoplus_{i=1}^{e(k)} \mathbb{Z}_{m(i,k)} \oplus \bigoplus_{i=1}^{o(k)} \mathbb{Z}_{n(i,k)}$$

where the $m(i,k)$ are even and the $n(i,k)$ are odd, $e(k)$ denotes the number of summands with even torsion in the kth homology group, $o(k)$ similarly denoting the number of summands with odd torsion, and β_k is the kth Betti number. The goal is to show that the free part of $H_1(M)$ is nontrivial (i.e. $\beta_1 > 0$) which will indeed show it is infinite ($\mathbb{Z}$ was infinite last time we checked).

We consider Poincaré Duality (PD) on M using $\mathbb{Z}_2$-orientability, since every manifold is $\mathbb{Z}_2$-orientable. Since $\mathbb{Z}_2$ is a field and $H_k(M;\mathbb{Z}_2)$ are finitely generated, we have

$$H^k(M;\mathbb{Z}_2) \cong \mathrm{Hom}_{\mathbb{Z}_2}(H_k(M;\mathbb{Z}_2),\mathbb{Z}_2) = (H_k(M;\mathbb{Z}_2))^* \cong H_k(M;\mathbb{Z}_2)$$

By PD, $H_1(M;\mathbb{Z}_2) \cong H^2(M;\mathbb{Z}_2)$ which in turn is isomorphic to $H_2(M;\mathbb{Z}_2)$ by the above. So using the Universal Coefficient Theorem (UCT) for H^1 and H^2 we have

$$H_1(M;\mathbb{Z}_2) \cong \mathrm{Ext}(H_0(M),\mathbb{Z}_2) \oplus \mathrm{Hom}(H_1(M),\mathbb{Z}_2) \tag{0.1}$$

$$\cong \mathrm{Ext}(H_1(M),\mathbb{Z}_2) \oplus \mathrm{Hom}(H_2(M),\mathbb{Z}_2) \tag{0.2}$$

Note that $\mathrm{Ext}(H_0(M),\mathbb{Z}_2) = 0$ since $H_0(M)$ is free, and since both Hom and Ext commute with direct sums we have

$$\mathrm{Ext}(H_k(M),\mathbb{Z}_2) \cong \mathbb{Z}_2^{e(k)} \tag{0.3}$$

becase $\mathbb{Z}_2/\ell\mathbb{Z}_2 = \mathbb{Z}_2$ when ℓ is even and $\mathbb{Z}_2/\ell\mathbb{Z}_2 = 0$ for ℓ odd; the free part always gets killed off by Ext. Similarly,

$$\mathrm{Hom}(H_k(M),\mathbb{Z}_2) \cong \mathbb{Z}_2^{\beta_k+e(k)} \tag{0.4}$$

because $\mathrm{Hom}(\mathbb{Z},\mathbb{Z}_2)$ and $\mathrm{Hom}(\mathbb{Z}_\ell,\mathbb{Z}_2) \cong \mathbb{Z}_2$ for ℓ even but $\mathrm{Hom}(\mathbb{Z}_\ell,\mathbb{Z}_2) = 0$ for ℓ odd. Combining all (0.1) and (0.4) we have $H_1(M;\mathbb{Z}_2) \cong \mathbb{Z}_2^{\beta_1+e(1)}$, and by (0.2), (0.3), and (0.4) we have $H_1(M;\mathbb{Z}_2) \cong \mathbb{Z}_2^{e(1)+(\beta_2+e(2))}$. So

$$\mathbb{Z}_2^{e(1)+\beta_2+e(2)} \cong \mathbb{Z}_2^{\beta_1+e(1)}$$

an isomorphism of abelian groups. Since $\mathbb{Z}_2$ is a “prime field” we have, by algebra, that this is also an isomorphism of $\mathbb{Z}_2$ vector spaces and hence $\beta_2 + e(1) + e(2) = \beta_1 + e(1)$ (i.e. dimension is invariant) so $\beta_1 = \beta_2 + e(2)$. Since $\beta_2 \geq 0$ if we show that $e(2) \geq 1$, this will show $\beta(1) \geq 1$ which is what we want.

We get this by noting that $H^3(M;\mathbb{Z}_2) \cong H_3(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$ since M is connected, compact, and $\mathbb{Z}_2$-orientable. Also $H_3(M) \cong 0$ as M is not ($\mathbb{Z}$-)orientable. Using the UCT one last time:

$$\mathbb{Z}_2 \cong H^3(M;\mathbb{Z}_2) \cong \mathrm{Ext}(H_2(M),\mathbb{Z}_2) \oplus \mathrm{Hom}(H_3(M),\mathbb{Z}_2) \cong \mathrm{Ext}(H_2(M),\mathbb{Z}_2)$$

But $\mathbb{Z}_2 \cong \mathrm{Ext}(H_2(M),\mathbb{Z}_2) \cong \mathbb{Z}_2^{e(2)}$ by (0.3) so $e(2) = 1$ and we are done. □