1. Introduction & Notation

This writeup will focus on the very last portion of the course, covering category-theoretic topics like natural transformations and models. And not just ordinary models – *acyclic* ones. Be forewarned, a lot of this will look like symbol-pushing, so before we start off here is a glossary of the terms that will be used:

$\mathcal{C}, \mathcal{D}$:	categories
Δ_q:	the standard q-simplex
δ_q:	the identity map on Δ_q
$\mathcal{M}$:	the models of $\mathcal{C}$
C:	an element in $\text{ob}(\mathcal{C})$
A:	an initial object in $\mathcal{C}$
Z:	a terminal object in $\mathcal{C}$
α, ω:	initial and terminal morphisms in $\mathcal{C}$
σ:	the flip map (sends $a \otimes b$ to $b \otimes a$)

2. Natural Transformations

For the duration of this writeup, the letters $\mathcal{C}$ and $\mathcal{D}$ will represent categories.

Definition 2.1 (Natural Transformations)**.** *Given two categories $\mathcal{C}$ and $\mathcal{D}$, and functors $F, F' : \mathcal{C} \to \mathcal{D}$, a* natural transformation *is a family of morphisms $p = \{p_X\} : F \to F'$ such that*

- *p carries each object A of $\mathcal{C}$ to a morphism $p_A : F(A) \to F'(A)$ of $\mathcal{D}$, and*
- *for any morphism $m : A \to B$ of $\mathcal{C}$, the following diagram commutes:*

$$\begin{array}{ccc} F(A) & \xrightarrow{p_A} & F'(A) \\ \downarrow{\scriptstyle F_m} & & \downarrow{\scriptstyle F'_m} \\ F(B) & \xrightarrow{p_B} & F'(B) \end{array}$$

2.1. **Further classification.** If all of the p_A turn out to be isomorphisms, then p is called a *natural isomorphism*. If the diagram above commutes for a *specific* morphism $\phi : A \to B$, then we say p is natural *over* ϕ. $\square$

2.2. **The boundary map.** One example we've seen repeatedly in this course is the boundary map ∂, which commutes with all inclusion maps $i : S_*(A) \to S_*(B)$ for spaces A and B. $\square$

2.3. **Singular complices.** Harper & Greenberg section (9.10) gives the following example of a natural transformation. Consider $S_q(X)$, the q-chains on X. If we fix a value in $S_q(\Delta_q)$ (denote it as $p(\delta_q)$), we can define the homomorphism $p_A : S_q(X) \to S_q(X)$ by

$$p_X : \sigma \to S_q(\sigma)p(\delta_q).$$

If we have a map $f : X \to Y$, then we'd like to show that the diagram below commutes:

$$\begin{array}{ccc} S_q(X) & \xrightarrow{p_X} & S_q(X) \\ \downarrow{\scriptstyle S_q(f)} & & \downarrow{\scriptstyle S_q(f)} \\ S_q(Y) & \xrightarrow{p_Y} & S_q(Y) \end{array}$$

For any given $\sigma \in S_q(X)$, we gots

$$p_Y(S_q(f)(\sigma)) = p_Y(f \circ \sigma) = S_q(f \circ \sigma)p(\delta_q) = S_q(f)\underbrace{S_q(\sigma)p(\delta_q)}_{p_X(\sigma)} = S_q(f)p_X(\sigma),$$

which means that p is a natural transformation. $\square$

3. Models

In order to arrive at the Acyclic Model Theorem, we need to first introduce the notion of models.

Definition 3.1 (Category with models)**.** *Given a category $\mathcal{C}$, let $\mathcal{M}$ be a subset of the objects in $\mathcal{C}$. Then $(\mathcal{C}, \mathcal{M})$ is a* category with models*, and $\mathcal{M}$ are the* models *of $\mathcal{C}$. (Whatever jokes or puns you're imagining right now most certainly have been made before.)*

The whole idea behind establishing models within categories is best likened to establishing bases for vector spaces. A vector space is just a quaint set of properties applied to a set of objects, but introduce the notion of basis and – *kablammo!* – you've got linear algebra.

Definition 3.2 (Free functor)**.** *If $\mathcal{C}$ is a category with models $\mathcal{M}$, then the functor $F : \mathcal{C} \to \mathcal{D}$ is* free *if*

- *FC is a free abelian group for every object $C \in ob(\mathcal{C})$, and*
- *there is an indexed set of models $\mathcal{X} = \{x_j \in FM_j | M_j \in \mathcal{M}, j \in J\}$ such that for every object C, the set $\{(F\sigma)(x_j) | x_j \in \mathcal{X}, \sigma : M_j \to C\}$ forms a basis for FC.*

There are a lot of symbols running around here, but essentially a free functor is one that admits a basis on the images FC. The simplest example we've used is on topological spaces:

3.1. **Topological Spaces.** Let $\mathcal{C}$ be the category of topological spaces, $\mathcal{M} = \{\Delta_k\}$, and $\mathcal{D}$ be the category of abelian groups. If F_k is the functor that takes the kth term of the singular complex, then we have $\mathcal{X} = \{\delta_k\}$.

Given any space X, the chains on X form a free abelian group with basis the k-simplices $\sigma : \Delta_k \to X$. Since $S_k(\sigma)(\delta_k) = \sigma$, we have

$$\{S_k(\sigma)(\delta_k) | \sigma : \Delta_k \to X\}$$

is a basis for $S_k(X)$. (The condition on $\mathcal{X}$ is trivially satisfied, since $|\mathcal{X}| = 1$.) Therefore S_k is free with a base $\{\Delta_k\}$. $\square$

Definition 3.3 (Acyclic functor)**.** *Let $\mathcal{D}$ be the category of chain complices, and $F : \mathcal{C} \to \mathcal{D}$ be a functor. An object $C \in ob(\mathcal{C})$ is called F-*acyclic *if $H_n(FC) = 0$ (with $H_0 = R$). If $\mathcal{C}$ is a category with models $\mathcal{M}$ and all the M_j are F-acyclic, then F is called an* acyclic functor*.*

3.2. **Convex spaces.** Let $(\mathcal{C}, \mathcal{M})$ be the category of topological spaces (with the models from the previous example), $\mathcal{D}$ be the category of chain complices, and $E : \mathcal{C} \to \mathcal{D}$ the functor sending X to $S_*(X \times I)$, where I is the unit interval.

Remember that every convex in $\mathbb{R}^n$ set has zero homology (except in degree 0). Since $\Delta_k \times I$ is convex, Δ_k is E-acyclic. Since all the models (that's right – *all* of them!) of $\mathcal{C}$ are E-acyclic, E is acyclic. $\square$

So the big deal with acyclic functors is that we can use them to construct natural chain maps (and possibly chain homotopies), as outlined in the following theorem:

Theorem 3.4 (Acyclic Model Theorem)**.** *Let $(\mathcal{C}, \mathcal{M})$ be a category with models, $\mathcal{D}$ be the category of augmented chain complices, and $E, F : \mathcal{C} \to \mathcal{D}$ be (non-negative) functors. Suppose for each $k \geq 0$ that F_k is free with base $\mathcal{M}_k \subset \mathcal{M}$ and all models of $\mathcal{M}$ are E-acyclic. Then...*

- *For all natural transformations $\phi : H_0F \to H_0E$, there is a natural chain map $\tau : F \to E$;*
- *Any two chain maps $\tau_1, \tau_2 : F \to E$ that are natural over ϕ are themselves naturally chain homotopic; and*
- *Given any natural isomorphism ϕ, every natural chain map $\tau : F \to E$ over ϕ is actually a natural chain isomorphism.*

This theorem is useful in that it constructs chain maps (and possibly chain homotopies) that are all natural. *Nice.*

4. Algebras of All Sorts

Before we introduce coalgebras and other rogue algebras (algebrae?), we'll need one small definition:

Definition 4.1 (Initial & terminal objects)**.** *Given a category $\mathcal{C}$, an object A is an* initial object *if, for every other object C in $\mathcal{C}$, there exists a unique morphism $\alpha : A \to C$. An object Z is a* terminal object *if, for every other object C in $\mathcal{C}$, there is a unique morphism $\omega : C \to Z$.*

Definition 4.2 (Coalgebra)**.** *Given a category $\mathcal{C}$ and initial object A, let C be an object in $\mathcal{C}$ and $\alpha : A \to C$ (α is uniquely determined). Let $\mu : C \to C \otimes C$, $h : C \to C$ and $\epsilon : C \to A$ be morphisms such that the following diagrams co-mmute:*

Co-Associativity:

$$\begin{array}{ccc} C & \xrightarrow{\mu} & C\otimes C \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle 1\otimes\mu} \\ C\otimes C & \xrightarrow{\mu\otimes 1} & C\otimes C\otimes C \end{array}$$

Co-Identity:

$$\begin{array}{ccccc} C\otimes A & \xleftarrow{1\otimes\epsilon} & C\otimes C & \xrightarrow{\epsilon\otimes 1} & A\otimes C \\ & \nwarrow & \uparrow{\scriptstyle\mu} & \nearrow & \\ & & C & & \end{array}$$

Co-Inverse:

$$\begin{array}{ccccc} C & \xleftarrow{1\otimes h} & C\otimes C & \xrightarrow{h\otimes 1} & C \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\mu} & & \uparrow{\scriptstyle\alpha} \\ A & \xleftarrow[\epsilon]{} & C & \xrightarrow[\epsilon]{} & A \end{array}$$

Then C is called a coalgebra*. If the flip map $\sigma : C\otimes C \to C \otimes C$ that sends $a \otimes b \to (-1)^{|a||b|} b \otimes a$ satisfies $\sigma \circ \mu = \mu$, then the coalgebra is said to be* cocommutative*, which is hard to say with a straight face.*

4.1. **Differences in definition.** For the purposes of our involvement in the area, we will be fine with the above definition. But! Different sources give varying definitions for coalgebras (or, as Rotman calls them, "co-group objects").

The big difference is the co-inverse diagram. Some sources include the co-inverse diagram with definition for coalgebra, and some include it separately (like how we have "rings" and "rings with unit"). □

4.2. **Abelian groups.** Let $\mathcal{C}$ be the category of abelian groups, G a non-identity group in $\mathcal{C}$, and $\mathbb{1}$ the one-element group ($\mathbb{1}$ is both an initial and terminal object in $\mathcal{C}$). Then, if μ is the diagonal map that sends $g \in G$ to (g, g), G is an coalgebra object. Since the diagonal map commutes with the flip map σ, G is consequently a cocommutative coalgebra. Cool! □

4.3. **Finitely generated free groups.** Let $\mathcal{C}$ be the category of groups, G be a free group with finitely many generators, $\mathbb{1}$ the one-element group and μ the diagonal map. Then G is a coalgebra. □

Now we move on to the notion of a Hopf algebra, a versatile tool that we will use to show that S^{2n} is not a topological group. First though, we need to introduce graded rings and co-rings:

Definition 4.3 (Graded rings & co-rings). *A ring R is a* graded ring *if it can be divvied up into additive subgroups R^n, $n \geq 0$, such that*

- $R = \sum_n R^n$, *and*
- $R^n R^m \subseteq R^{n+m}$

A graded abelian group $G = \sum_n G^n$ with a homomorphism $\phi : G \to G \otimes G$ is called a graded co-ring *if*

$$\phi(G^n) \subseteq \sum_{i+j=n} G^i \otimes G^j$$

4.4. **Cochains on X.** For any space X and ring R, the cochains $S^*(X, R)$ is a graded ring under the cup product $\cup$. This comes from the very definition of the cup product, which takes $\phi_1 \in S^n(X, R)$ and $\phi_2 \in S^m(X, R)$ and defines $(\sigma, \phi_1 \cup \phi_2) = (\sigma\lambda_n, \phi_1)(\sigma\mu_m, \phi_2)$, where the λ_k and μ_k are the front face and back face maps, respectively. □

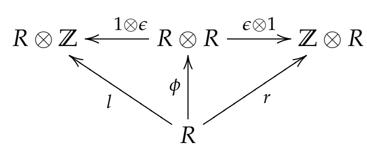
4.5. **Polynomial rings.** For any commutative ring R, the polynomial ring $R[x]$ is a graded ring. □

4.6. **Exterior algebras.** For an R-module M, the set of wedges of elements of M, $\sum_{n\geq 0} \bigwedge_n M$ is a graded ring if we set $(x_1 \wedge \cdots \wedge x_p) \cdot (y_1 \wedge \cdots \wedge y_q) = x_1 \wedge \cdots \wedge x_p \wedge y_1 \wedge \cdots \wedge y_q$. □

Definition 4.4 (Hopf algebra). *A graded abelian group $G (= \sum_{n\geq 0} G^n)$ is a* Hopf algebra over $\mathbb{Z}$ *if it satisfies all of the following:*

- *G is a graded ring;*
- *G is a graded co-ring; and*
- *the co-ring homomorphism $\phi : G \to G \otimes G$ is also a graded ring homomorphism.*

4.7. **Co-units and connectedness.** The term *co-unit* has come up a bunch in class, and is a tool we need to eventually talk about Hopf algebras of $H*(X)$. For a graded co-ring R, the homomorphism $\epsilon : R \to \mathbb{Z}$ is a *co-unit* if it makes the following diagram commute:



Here, the homomorphisms l and r are the quasi-inclusions given by $l : x \to x \otimes 1$ and $r : x \to 1 \otimes x$.

The big classification on Hopf algebras that we will concern ourselves with is the class of *connected* Hopf algebras. If

- R^0 is infinite cyclic (i.e., $\cong \mathbb{Z}$) with generator e, and
- the map $\epsilon : R \to \mathbb{Z}$, which sends $e \to 1$ and $r^n \to 0$ for any $n \geq 1$, is a co-unit,

then R is a *connected* Hopf algebra. □

4.8. **Path-connected H-spaces.** For any path-connected H-space X whose homology groups are all finitely generated free abelian groups, $H^*(X)$ is a connected Hopf algebra over $\mathbb{Z}$. This is Theorem 12.42 in Rotman (p. 416), and the proof is very longwinded. The gist of it is quite nice, though: if a space X has a "nice" homology structure but $H^*(X)$ is *not* a connected Hopf algebra, then we can immediately conclude that X is not an H-space, and therefore not a topological group. □

4.9. **The even sphere is not a topological group.** We know already that $H^*(S^{2n})$ only has nonzero pieces in degree 0 and $2n$, both infinite cyclic. Let the generators of H^0 and H^{2n} be e and x, respectively. To show that S^{2n} is not a topological group, we need only show it isn't a connected Hopf algebra. So, for kicks, let's assume it *is*.

The comultiplication morphism $\mu : H^* \to H^* \otimes H^*$ is a map of graded rings. Since x is a degree-$2n$ element, we have

$$\mu(x) = re \otimes sx + ux \otimes ve,$$

where $r, s, u, v \in \mathbb{Z}$. Let $\epsilon : H^* \to \mathbb{Z}$ be the co-unit homomorphism. Then

$$(\epsilon \otimes 1)\mu(x) = x \otimes 1 \quad \text{and} \quad (1 \otimes \epsilon)\mu(x) = 1 \otimes x.$$

Okay, so the isomorphisms from $H^0 \otimes H^{2n}$ and $H^{2n} \otimes H^0$ onto $\mathbb{Z}$ imply $rs = uv = 1$, meaning $r = s = \pm 1 = u = v$. Since we can commute constants across the $\otimes$, we have $\mu(x) = e \otimes x + x \otimes e$. Now, since $x \cup x$ has degree $4n$, it must be 0, and so $\mu(x \cup x) = 0$ as well. But, using the multiplicativeness of μ over cup products, we get

$$\begin{aligned} 0 &= \mu(x)\mu(x) \\ &= (e \otimes x + x \otimes e)(e \otimes x + x \otimes e) \\ &= (e \otimes x)^2 + (x \otimes e)^2 + (x \otimes e)(e \otimes x) + (e \otimes x)(x \otimes e) \\ &= (e \otimes (x \cup x)) + ((x \cup x) \otimes e) + (x \otimes x) + (-1)^{(2n)^2}(x \otimes x) \\ &= 0 + 0 + 2(x \otimes x) \\ &\Rightarrow \boxed{x \otimes x = 0} \end{aligned}$$

But for a free abelian group G and torsion-free group A, we have shown that if $g \in G$ is nonzero, then $a \otimes g \neq 0$ for all $a \neq 0$ in A. Since $H^{2n} \cong \mathbb{Z}$ is both free abelian and torsion free, we have $x \otimes x \neq 0$, a contradiction. Tracing all the way back, we have $H^*(S^{2n})$ is not an H-space, and consequently not a topological group.