Polynomial Matrix Inequality and Semidefinite Representation

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Abstract

Consider a convex set $S = \{x \in \mathcal{D} : G(x) \succeq 0\}$ where $G(x)$ is an $m \times m$ symmetric matrix whose every entry is a polynomial or rational function, $\mathcal{D} \subseteq \mathbb{R}^n$ is a domain where $G(x)$ is defined, and $G(x) \succeq 0$ means $G(x)$ is positive semidefinite. The set S is called semidefinite programming (SDP) representable or just semidefinite representable if it equals the projection of a higher dimensional set which is defined by a linear matrix inequality (LMI). This paper studies sufficient conditions guaranteeing semidefinite representability of S. We prove that S is semidefinite representable in the following cases: (i) $\mathcal{D} = \mathbb{R}^n$, $G(x)$ is a matrix polynomial and matrix sos-concave; (ii) $\mathcal{D}$ is compact convex, $G(x)$ is a matrix polynomial and strictly matrix concave on $\mathcal{D}$; (iii) $G(x)$ is a matrix rational function and q-module matrix concave on $\mathcal{D}$. Explicit constructions of SDP representations are given. Some examples are illustrated.

Keywords convex sets, linear matrix inequality, matrix concavity, polynomial matrix inequality, rational function, semidefinite programming, sum of squares

AMS subject classification (2000) 90C22, 90C25

1 Introduction

Suppose S is a convex set in $\mathbb{R}^n$ given as

$$S = \{x \in \mathcal{D} : G(x) \succeq 0\}. \tag{1.1}$$

Here $\mathcal{D} \subseteq \mathbb{R}^n$ is a domain, and $G(x)$ is a $m \times m$ symmetric matrix polynomial, that is, every entry of $G(x)$ is a polynomial in x. The notation $A \succeq 0$ (resp. $A \succ 0$) means the matrix A is positive semidefinite (resp. definite). Suppose $G(x)$ has total degree $2d$ and

$$G(x) = \sum_{\alpha \in \mathbb{N}^n :\, \alpha_1 + \cdots + \alpha_n \leq 2d} G_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}. \tag{1.2}$$

The G_α are constant symmetric matrices. The $G(x) \succeq 0$ is called a polynomial matrix inequality (PMI). When $G(x)$ is linear, optimizing a linear functional over S becomes a standard semidefinite programming (SDP) problem. SDP is a very nice convex optimization,

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has many attractive properties, and can be solved efficiently by numerical methods. We refer to [17, 24, 26, 27]. There are standard packages like SDPA [4], SDPT3 [25] and SeDuMi [23] for solving SDP problems. It would be a big advantage if an optimization problem can be formulated in SDP form. So we are very interested in knowing when and how the set S can be represented by SDP.

One elementary approach for this representation problem is to find symmetric matrices $A_1, \ldots, A_n$ such that

$$S = \{x \in \mathbb{R}^n : A_0 + A_1 x_1 + \cdots + A_n x_n \succeq 0\}.$$

If such A_i exist, we say S has a linear matrix inequality (LMI) representation and S is LMI representable. Unfortunately, not every convex set in $\mathbb{R}^n$ is LMI representable. For instance, the convex set $\{x \in \mathbb{R}^2 : 1 - x_1^4 - x_2^4 \geq 0\}$ is not LMI representable, as proved by Helton and Vinnikov [9]. Therefore, we are more interested in finding a lifted LMI representation, that is, in addition to A_i, finding symmetric matrices $B_1, \ldots, B_N$ such that

$$S = \left\{ x \in \mathbb{R}^n : \exists\, y \in \mathbb{R}^N, A_0 + \sum_{i=1}^n A_i x_i + \sum_{j=1}^N B_j y_j \succeq 0 \right\}. \tag{1.3}$$

If such matrices A_i and B_j exist, we say S is semidefinite programming (SDP) representable or just semidefinite representable, and (1.3) is a lifted LMI or semidefinite representation for S. The variables y_j are called lifting variables. Nesterov and Nemirovski [17], Ben-Tal and Nemirovski [2], and Nemirovski [18] gave collections of convex sets that are SDP representable. Obviously, to have a lifted LMI, a convex set must be convex and semialgebraic, i.e., it can be defined by a boolean combination of scalar polynomial inequalities. However, it is unclear whether every convex semialgebraic set has a lifted LMI or not.

When $G(x)$ is diagonal, i.e., S is defined by scalar polynomial inequalities, there is some work on the semidefinite representability of S. Parrilo [21] constructed lifted LMIs for planar convex sets whose boundaries are rational planar curves of genus zero. Lasserre constructed lifted LMIs for convex semialgebraic sets satisfying certain conditions like bounded degree representation (BDR) [12, 13]. Their constructions use moments and sum of squares techniques. In [6], Helton and Nie proved sufficient conditions like sos-convexity and strict convexity, which validate lifted LMIs from moment type constructions. Later in [7], they further proved every compact convex semialgebraic set is SDP representable if its boundary is nonsingular and positively curved. Recent work in this area can be found in [1, 5, 8, 10, 11, 14, 15, 19, 20].

We might consider to apply the existing results for the case of scalar polynomial inequalities like in [6, 7, 12, 13, 19] to the case of matrix polynomial inequalities. Note

$$S = \Big\{ x \in \mathcal{D} : \, p_I(x) \geq 0 \quad \forall\, I \subset \{1, 2, \ldots, m\} \Big\}.$$

Here polynomials $p_I(x)$ are principal minors of matrix $G(x)$ with row (or column) index I. So all the results in [6, 7, 12, 13] can be applied to study the semidefinite representability of S by investigating the properties of principal minors $g_I(x)$. If every $g_I(x)$ is sos-concave, or S is compact convex and its boundary is nonsingular and positively curved, then it is semidefinite representable as shown in [6, 7]. However, these conditions are based on the principal minors of $G(x)$ or the geometry of S, and hence are difficult to be applicable in many situations, especially when the matrix $G(x)$ has big dimensions. The conditions directly on $G(x)$ are preferable in practical applications. The motivation of this paper is to construct SDP representations for S and prove sufficient conditions directly on $G(x)$ justifying them.

In some particular applications, $G(x)$ might be a matrix rational function, i.e., every entry of $G(x)$ is a rational function. This is often the case in control theory. When $G(x)$ is a scalar rational function, the author in [19] studied SDP representability of S. In [19], explicit constructions of lifted LMIs are given, and sufficient conditions validating them are proved. In this paper, we will construct lifted LMIs for the more general case that $G(x)$ is a matrix rational function, and prove sufficient conditions justifying them.

This paper is organized as follows. Section 2 discusses the semidefinite representation of S when $\mathcal{D} = \mathbb{R}^n$, and $G(x)$ is a matrix polynomial and matrix sos-concave. Section 3 discusses the semidefinite representation of S when $\mathcal{D}$ is a compact convex domain, and $G(x)$ is a matrix polynomial and strictly matrix concave on $\mathcal{D}$. The case that $G(x)$ is a matrix rational function and is q-module matrix concave over $\mathcal{D}$ will be discussed in Section 4.

Notations. The symbol $\mathbb{N}$ (resp., $\mathbb{R}$) denotes the set of nonnegative integers (resp., real numbers). For any $t \in \mathbb{R}$, $\lceil t \rceil$ denotes the smallest integer not smaller than t. The $\mathbb{R}^n_+$ denotes the nonnegative orthant. For $x \in \mathbb{R}^n$, x_i denotes the i-th component of x, that is, $x = (x_1, \dots, x_n)$. For $\alpha \in \mathbb{N}^n$, denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$, x^α denotes $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. For $\alpha, \beta \in \mathbb{N}^n$, denote $\alpha \leq \beta$ if every $\alpha_i \leq \beta_i$. The symbol $\mathbb{N}_{\leq k}$ denotes the multi-index set $\{\alpha \in \mathbb{N}^n : |\alpha| \leq k\}$. For every integer $i \geq 0$, e_i denotes the i-th standard unit vector. The $[x]_d$ denotes the vector of all monomials having degrees at most d and ordered gradedly alphabetically, that is,

$$[x]_d^T = [\,1 \quad x_1 \quad \cdots \quad x_n \quad x_1^2 \quad x_1x_2 \quad \cdots x_n^2 \quad \cdots \quad x_1^d \quad x_1^{d-1}x_2 \quad \cdots \quad x_n^d\,].$$

A polynomial $p(x)$ is said to be a sum of squares (sos) if there exist finitely many polynomials $q_i(x)$ such that $p(x) = \sum q_i(x)^2$. A matrix polynomial $H(x)$ is called sos if there is a matrix polynomial $F(x)$ such that $H(x) = F(x)^T F(x)$. For a set S, $int(S)$ denotes its interior, and ∂S denotes its boundary. For $u \in \mathbb{R}^N$, $\|u\|_2$ denotes the standard Euclidean norm. For a matrix X, X^T denotes its transpose, $\|X\|_F$ denotes the Frobinus norm of X, i.e., $\|X\|_F = \sqrt{Trace(X^T X)}$, and $\|X\|_2$ denotes the standard operator 2-norm of X. The symbol $\bullet$ denotes the standard Frobinus inner product of matrix spaces, and I_N denotes the $N \times N$ identity matrix. For a function $f(x)$, $\mathcal{Z}(f) = \{x \in \mathbb{R}^n : f(x) = 0\}$, $\nabla_x f(x)$ denotes its gradient with respect to x, and $\nabla_{xx} f(x)$ denotes its Hessian with respect to x.

2 Matrix sos-concavity

In this section, assume the domain $\mathcal{D} = \mathbb{R}^n$ is the whole space and $G(x)$ is an $m \times m$ symmetric matrix polynomial of degree $2d$. We will first construct an SDP relaxation for S using moments, and then prove it is a lifted LMI when $G(x)$ satisfies certain conditions.

A natural SDP relaxation of S can be obtained through moments. Define linear matrix pencils $\mathcal{G}(y)$ and $\mathcal{A}_d(y)$ as

$$\mathcal{G}(y) = \sum_{\alpha \in \mathbb{N}_{\leq 2d}} y_\alpha G_\alpha, \quad \mathcal{A}_d(y) = \sum_{\alpha \in \mathbb{N}_{\leq 2d}} y_\alpha A_\alpha^{(d)},$$

where G_α are from (1.2) and $A_\alpha^{(d)}$ are such that

$$[x]_d[x]_d^T = \sum_{\alpha \in \mathbb{N}_{\leq 2d}} x^\alpha A_\alpha^{(d)}. \tag{2.1}$$

Since $S = \left\{x \in \mathbb{R}^n : G(x) \succeq 0,\, [x]_d[x]_d^T \succeq 0\right\}$, we know

$$S = \{(y_{e_1}, \ldots, y_{e_n}) \in \mathbb{R}^n : \exists\, x \in \mathbb{R}^n,\, y_\alpha = x^\alpha\ \forall\, \alpha \in \mathbb{N}_{\leq 2d},\, \mathcal{G}(y) \succeq 0,\, \mathcal{A}_d(y) \succeq 0\}.$$

Here each e_i denote the i-th standard unit vector whose only nonzero entry is one at index i. If the condition $y_\alpha = x^\alpha$ is removed in the above, then S is a subset of

$$L = \left\{x \in \mathbb{R}^n : \exists\, y \in \mathbb{R}^{\binom{n+2d}{2d}},\, y_0 = 1, x_1 = y_{e_1}, \ldots, x_n = y_{e_n},\, \mathcal{G}(y) \succeq 0,\, \mathcal{A}_d(y) \succeq 0\right\}. \tag{2.2}$$

So $S \subseteq L$. Does $S = L$? What conditions make $S = L$? This section will look for sufficient conditions ensuring $S = L$.

The matrix-valued function $G(x)$ is called *matrix concave* if for all $u, v \in \mathbb{R}^n$ and $0 \leq \theta \leq 1$

$$G(\theta u + (1-\theta)v) \succeq \theta G(u) + (1-\theta)G(v).$$

The matrix concavity of $G(x)$ is equivalent to

$$-\nabla_{xx}(\xi^T G(x)\xi) \succeq 0 \quad \forall\, \xi \in \mathbb{R}^m,\, \forall\, x \in \mathbb{R}^n.$$

Generally it is difficult to check the matrix concavity of matrix polynomials. Even for the simple case that they are quadratic, the problem is already NP-hard, as shown below.

Proposition 2.1. *It is NP-hard to check the matrix concavity of quadratic matrix polynomials.*

Proof. Let $\overline{m} = \frac{1}{2}m(m+1)$. For any symmetric matrices $A_1, \ldots, A_{\overline{m}} \in \mathbb{R}^{m \times m}$ and $B_1, \ldots, B_{\overline{m}} \in \mathbb{R}^{n \times n}$, define matrix polynomial

$$G(x) = -\frac{1}{2}\sum_{i=1}^{\overline{m}} (x^T B_i x) A_i.$$

Then we have

$$-\nabla_{xx}(\xi^T G(x)\xi) = \sum_{i=1}^{\overline{m}} (\xi^T A_i \xi) B_i.$$

So $G(x)$ is matrix concave if and only if the following bi-quadratic form

$$\sum_{i=1}^{\overline{m}} (\xi^T A_i \xi)(x^T B_i x)$$

is always nonnegative. It has been proven in [16] that it is NP-hard to check the nonnegativity of bi-quadratic forms. Therefore, it must also be NP-hard to check the matrix concavity of quadratic $G(x)$. □

A stronger but easier checkable condition than matrix concavity is the so called matrix sos-concavity. We say $G(x)$ is *matrix sos-concave* if for every $\xi \in \mathbb{R}^m$, there exists a matrix polynomial $F_\xi(x)$ in x such that

$$-\nabla_{xx}(\xi^T G(x)\xi) = F_\xi(x)^T F_\xi(x). \tag{2.3}$$

The above $F_\xi(x)$ has n columns but its number of rows might be different from n, and its coefficients of x^α depend on ξ.

Theorem 2.2. *Suppose $G(\tilde{x}) \succ 0$ for some $\tilde{x}$. If $G(x)$ is matrix sos-concave, then $S = L$.*

Proof. We have already seen $S \subseteq L$, so it suffices to prove the reverse containment. Otherwise suppose $L \neq S$, then there must exist a point $\hat{x} \in L/S$. Since S is closed and convex, by Hahn-Banach Theorem, there exists a supporting hyperplane $\mathcal{H} = \{x \in \mathbb{R}^n : a^T x \geq b\} \supseteq S$ such that $a^T u = b$ for some $u \in S$ and $a^T \hat{x} < b$. Consider the linear optimization

$$\min_{x \in \mathbb{R}^n} a^T x \quad \text{subject to} \quad G(x) \succeq 0. \tag{2.4}$$

Clearly u is a minimizer and b is the optimal value. The optimization (2.4) is convex. The existence of $\tilde{x}$ with $G(\tilde{x}) \succ 0$, i.e., the Slater's condition holds, implies there exists a matrix Lagrange multiplier $\Lambda \succeq 0$ such that

$$\Lambda \bullet G(u) = 0, \quad a = \nabla_x(\Lambda \bullet G(x))\Big|_{x=u}.$$

The value and gradient of $a^T x - \Lambda \bullet G(x) - b$ vanish at u. Then, by the Taylor expansion at u, we have

$$a^T x - \Lambda \bullet G(x) - b = (x-u)^T \left(\int_0^1 \int_0^t -\nabla_{xx}(\Lambda \bullet G(u + s(x-u))\, ds\, dt \right)(x-u).$$

Since $\Lambda \succeq 0$, there exist vectors $\lambda^{(k)}$ such that $\Lambda = \sum_{k=1}^K \lambda^{(k)}(\lambda^{(k)})^T$. So we have

$$a^T x - \Lambda \bullet G(x) - b =$$
$$\sum_{k=1}^K (x-u)^T \left(\int_0^1 \int_0^t -\nabla_{xx}((\lambda^{(k)})^T G(u + s(x-u))\lambda^{(k)}\, ds\, dt \right)(x-u).$$

Since $G(x)$ is matrix sos-concave, by Lemma 7 in [6], we know each summand in the above must be sos. Thus $a^T x - \Lambda \bullet G(x) - b$ must also be an sos polynomial of degree $2d$. So there exists a symmetric matrix $W \succeq 0$ such that the identity

$$a^T x - \Lambda \bullet G(x) - b = [x]_d^T W [x]_d = W \bullet \left([x]_d [x]_d^T\right)$$

holds. By definition of matrices $A_\alpha^{(d)}$ in (2.1), we have

$$a^T x - \Lambda \bullet G(x) - b = W \bullet \left(\sum_{\alpha \in \mathbb{N}_{\leq 2d}} x^\alpha A_\alpha^{(d)} \right).$$

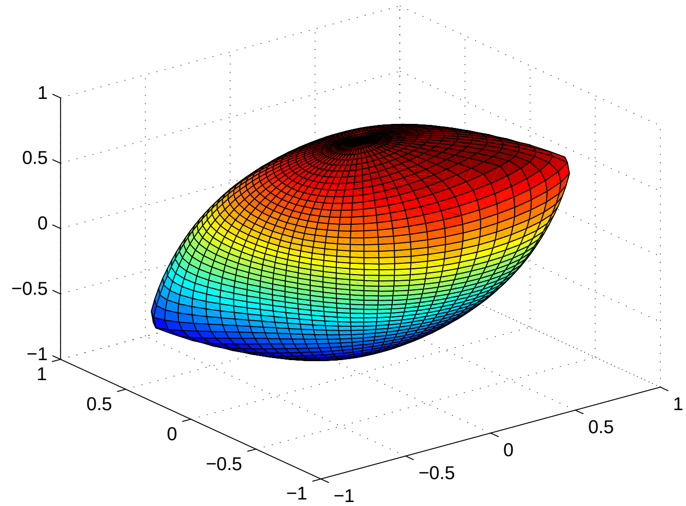

Figure 1: The drawn body is the convex set in Example 2.3.

Since $\hat{x} \in L$, there exists $\hat{y}$ such that $\hat{x} = (\hat{y}_{e_1}, \ldots, \hat{y}_{e_n})$, $\mathcal{G}(\hat{y}) \succeq 0$ and $\mathcal{A}_d(\hat{y}) \succeq 0$. So in the above identity, if we replace every monomial $\hat{x}^\alpha$ by $\hat{y}_\alpha$, then

$$a^T\hat{x} - \Lambda \bullet \mathcal{G}(\hat{y}) - b = W \bullet \left(\sum_{|\alpha| \le 2d} \hat{y}_\alpha A_\alpha^{(d)} \right) = W \bullet \mathcal{A}_d(\hat{y}),$$

or equivalently

$$a^T\hat{x} - b = \Lambda \bullet \mathcal{G}(\hat{y}) + W \bullet \mathcal{A}_d(\hat{y}).$$

Since $\Lambda, \mathcal{G}(\hat{y}), W, \mathcal{A}_d(\hat{y}) \succeq 0$, we must have $a^T\hat{x} - b \ge 0$, which contradicts the previous assertion that $a^T\hat{x} - b < 0$. So $S = L$. □

Example 2.3. Consider the set $\{x \in \mathbb{R}^3 : G(x) \succeq 0\}$ where

$$G(x) = \begin{bmatrix} 2 - x_1^2 - 2x_3^2 & 1 + x_1x_2 & x_1x_3 \\ 1 + x_1x_2 & 2 - x_2^2 - 2x_1^2 & 1 + x_2x_3 \\ x_1x_3 & 1 + x_2x_3 & 2 - x_3^2 - 2x_2^2 \end{bmatrix}.$$

The plot of this set is in Figure 1. The Hessian $-\nabla_{xx}(\xi^T G(x)\xi)$ is positive semidefinite for all $\xi \in \mathbb{R}^3$. This is because

$$-\nabla_{xx}(\xi^T G(x)\xi) = \begin{bmatrix} \xi_1^2 + 2\xi_3^2 & -\xi_1\xi_2 & -\xi_1\xi_3 \\ -\xi_1\xi_2 & \xi_2^2 + 2\xi_1^2 & -\xi_2\xi_3 \\ -\xi_1\xi_3 & -\xi_2\xi_3 & \xi_3^2 + 2\xi_2^2 \end{bmatrix} \succeq 0 \quad \forall \xi \in \mathbb{R}^3,$$

which is due the fact that the bi-quadratic form $x^T\big(-\nabla_{xx}(\xi^T G(x)\xi)\big)x$ in (x, ξ)

$$x_1^2\xi_1^2 + x_2^2\xi_2^2 + x_3^2\xi_3^2 + 2(x_1^2\xi_2^2 + x_2^2\xi_3^2 + x_3^2\xi_1^2) - 2(x_1x_2\xi_1\xi_2 + x_1x_3\xi_1\xi_3 + x_2x_3\xi_2\xi_3)$$

is always nonnegative, as shown by Choi [3]. So this set is convex, and by Theorem 2.2 a lifted LMI for it is

$$\left\{ \begin{array}{l} x \in \mathbb{R}^3 : \exists\, y_\alpha \, (\alpha \in \mathbb{N}_{\leq 2}) \quad \text{such that} \\ \begin{bmatrix} 2 - y_{200} - 2y_{002} & 1 + y_{110} & y_{101} \\ 1 + y_{110} & 2 - y_{020} - 2y_{200} & 1 + y_{011} \\ y_{101} & 1 + y_{011} & 2 - y_{002} - 2y_{020} \end{bmatrix} \succeq 0, \end{array} \begin{bmatrix} 1 & x_1 & x_2 & x_3 \\ x_1 & y_{200} & y_{110} & y_{101} \\ x_2 & y_{110} & y_{020} & y_{011} \\ x_3 & y_{101} & y_{011} & y_{002} \end{bmatrix} \succeq 0 \right\}.$$

There are in total 6 lifting variables. The plot of the x in the above coincides with the convex body of Figure 1, which confirms this lifted LMI is valid. □

The matrix sos-concavity condition requires us to check the Hessian

$$-\nabla_{xx}(\xi^T G(x)\xi)$$

is sos for every $\xi \in \mathbb{R}^m$. This is almost impossible since there are uncountably many cases. However, a stronger condition called *uniformly matrix sos-concave* is

$$-\nabla_{xx}(\xi^T G(x)\xi) = F(\xi, x)^T F(\xi, x),$$

where $F(\xi, x)$ is now a matrix polynomial in joint variables (ξ, x). It is easier to check. The uniformly matrix sos-concavity can be verified by solving a single semidefinite programming feasibility problem. Clearly the following is a consequence of Theorem 2.2.

Corollary 2.4. *Suppose $G(\tilde{x}) \succ 0$ for some $\tilde{x}$. If $G(x)$ is uniformly matrix sos-concave, then $S = L$.*

It should be pointed out that when $G(x)$ is matrix sos-concave, it is not necessarily that $G(x)$ is uniformly matrix sos-concave. For a counterexample, consider the $G(x)$ defined in Example 2.3, the quadratic matrix polynomial $-\nabla_{xx}(\xi^T G(x)\xi)$ there is not sos in ξ, as shown by Choi [3]. Now let us see an example of uniformly matrix sos-concave $G(x)$.

Example 2.5. Consider the set $\{x \in \mathbb{R}^2 : G(x) \succeq 0\}$ where

$$G(x) = \begin{bmatrix} 2 - 2x_1^4 - 4x_1^2x_2^2 - 2x_4^4 & 3 - x_1^3x_2 - x_1x_2^3 \\ 3 - x_1^3x_2 - x_1x_2^3 & 5 - x_1^4 - 4x_1^2x_2^2 - x_4^4 \end{bmatrix}.$$

A plot of this set is in Figure 2. The above $G(x)$ is uniformly matrix sos-concave because

$$-\nabla_{xx}(\xi^T G(x)\xi) = H_1 + H_2 + H_3 + H_4,$$

$$H_1 = 2\begin{bmatrix} 2\xi_1x_1 + \xi_2x_2 \\ 2\xi_1x_2 + \xi_2x_1 \end{bmatrix}\begin{bmatrix} 2\xi_1x_1 + \xi_2x_2 \\ 2\xi_1x_2 + \xi_2x_1 \end{bmatrix}^T, \quad H_2 = 8(\xi_1^2 + \xi_2^2)\begin{bmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{bmatrix},$$

$$H_3 = 2\begin{bmatrix} \xi_1x_1 & \xi_2x_2 & \xi_2x_1 \\ \xi_2x_1 & \xi_1x_2 & \xi_2x_2 \end{bmatrix}\begin{bmatrix} \xi_1x_1 & \xi_2x_2 & \xi_2x_1 \\ \xi_2x_1 & \xi_1x_2 & \xi_2x_2 \end{bmatrix}^T,$$

$$H_4 = 2\left(\left((\xi^Tx)^2 + \xi_2^2x_1^2\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \xi_1^2\begin{bmatrix} 2x_1^2 + 4x_2^2 & 0 \\ 0 & 3x_1^2 + 3x_2^2 \end{bmatrix}\right).$$

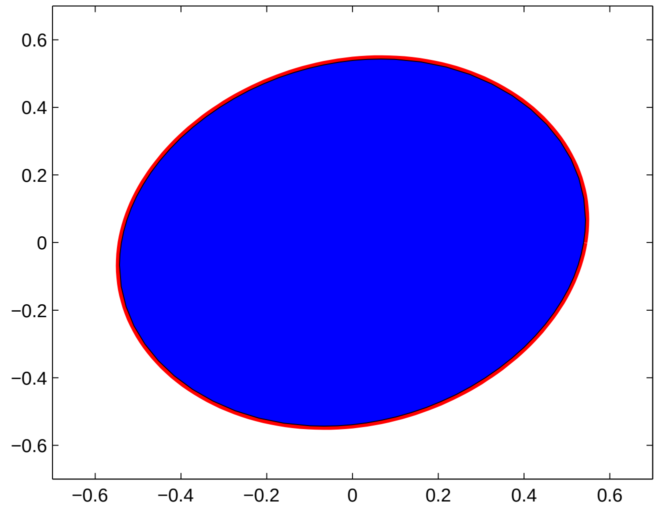

Figure 2: The shaded area is the convex set in Example 2.5, and the thick curve is its boundary.

So this set is convex, and by Corollary 2.4 a lifted LMI for it is

$$\left\{ \begin{array}{l} x \in \mathbb{R}^2 : \exists\, y_{ij}\,(0 \le i,j \le 4) \text{ such that} \\ \\ \begin{bmatrix} 2-2(y_{40}+2y_{22}+y_{04}) & 3-(y_{31}+y_{13}) \\ 3-(y_{31}+y_{13}) & 5-(y_{40}+3y_{22}+y_{04}) \end{bmatrix} \succeq 0, \end{array} \begin{bmatrix} 1 & x_1 & x_2 & y_{20} & y_{11} & y_{02} \\ x_1 & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ x_2 & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{bmatrix} \succeq 0 \right\}.$$

There are in total 12 lifting variables y_{ij}. The plot of x in the above coincides with the shaded area of Figure 2, which confirms this lifted LMI is valid. □

3 Strict matrix concavity

In this section, assume $S = \{x \in \mathcal{D} : G(x) \succeq 0\}$ where

$$\mathcal{D} = \{x \in \mathbb{R}^n : g_1(x) \ge 0, \ldots, g_m(x) \ge 0\}$$

is a domain defined by multivariate polynomials $g_1(x), \ldots, g_m(x)$. When $\mathcal{D}$ is compact convex and $G(x)$ is strictly matrix concave on $\mathcal{D}$, we will show S is semidefinite representable, and a lifted LMI for it can be explicitly constructed.

Like in the previous section, a natural SDP relaxation of S can be constructed by using moments. Let $g_0(x) = 1$, and set $d = \max\{\deg(G(x))/2, \lceil \deg(g_k)/2 \rceil, k = 0, 1, \ldots, m\}$. For every integer $N \ge d$ and $k = 0, \ldots, m$, define symmetric matrices $B_{k,\beta}^{(N)}$ such that

$$g_k(x)[x]_{N-d_k}[x]_{N-d_k}^T = \sum_{\beta \in \mathbb{N}_{\le 2N}} x^\beta B_{k,\beta}^{(N)}. \tag{3.1}$$

This determines $B_{k,\beta}^{(N)}$ uniquely. Then define linear matrix pencils $B_k^{(N)}(y)$ as

$$B_k^{(N)}(y) = \sum_{\beta \in \mathbb{N}_{\leq 2N}} y_\beta B_{k,\beta}^{(N)}, \quad k = 0, 1, \dots, m.$$

Clearly S can be equivalently described as

$$S = \left\{ (y_{e_1}, \dots, y_{e_n}) \in \mathbb{R}^n : \begin{array}{l} \exists x \in \mathbb{R}^n, y_\alpha = x^\alpha \quad \forall \, \alpha \in \mathbb{N}_{\leq 2N}, \\ \mathcal{G}(y) \succeq 0, \, B_k^{(N)}(y) \succeq 0, k = 0, \dots, m \end{array} \right\}.$$

If the condition $y_\alpha = x^\alpha$ is removed in the above, then S is contained in the set

$$L_N = \left\{ x \in \mathbb{R}^n : \exists y \in \mathbb{R}^{\binom{n+2N}{n}}, \begin{array}{l} y_0 = 1, x_1 = y_{e_1}, \dots, x_n = y_{e_n}, \\ \mathcal{G}(y) \succeq 0, \, B_k^{(N)}(y) \succeq 0, k = 0, \dots, m \end{array} \right\}. \tag{3.2}$$

So we have $S \subseteq L_N$ for every $N \geq d$. Clearly $L_{N+1} \subseteq L_N$, because L_{N+1} is a restriction of L_N. Thus it holds the nesting containment relation:

$$L_d \supseteq \cdots \supseteq L_N \supseteq L_{N+1} \supseteq \cdots \supseteq S.$$

Does there exist a finite N such that $S = L_N$? What conditions on S make it true? This is the main topic of this section.

Semidefinite representation of S is closely related to linear functionals nonnegative on S. For a given $0 \neq \ell \in \mathbb{R}^n$, consider the linear optimization

$$\min_{x \in \mathcal{D}} \quad \ell^T x \quad \text{subject to} \quad G(x) \succeq 0. \tag{3.3}$$

It always has a minimizer $u \in \partial S$ when S is compact. If $S \subseteq int(\mathcal{D})$, then $u \in int(\mathcal{D})$, and hence the constraint $x \in \mathcal{D}$ in (3.3) is not active. If further there exists $\tilde{x} \in \mathcal{D}$ such that $G(\tilde{x}) \succ 0$, i.e., the *Slater's condition* holds, and $G(x)$ is matrix concave in $\mathcal{D}$, then there exists a matrix Lagrange multiplier $0 \preceq \Lambda \in \mathbb{R}^{m \times m}$ such that

$$\Lambda \bullet G(u) = 0, \quad \ell = \nabla_x (\Lambda \bullet G(x))\big|_{x=u}. \tag{3.4}$$

Thus, by its Taylor expansion of at u, we know $\ell^T(x-u) - \Lambda \bullet G(x)$ equals

$$(x-u)^T \cdot \underbrace{\left(\int_0^1 \int_0^t -\nabla_{xx} (\Lambda \bullet G(u + s(x-u)) \,) \, ds \, dt \right)}_{H(u,x)} \cdot (x-u). \tag{3.5}$$

If the above matrix polynomial $H(u,x)$ has a weighted sos representation in terms of $G(x)$ and $g_i(x)$, then we can also get a similar one for $\ell^T(x-u)$. For this purpose, we need some assumptions on $\mathcal{D}$ and $G(x)$.

Assumption 3.1. *$G(x)$ is matrix concave on $\mathcal{D}$, and $G(x)$ satisfies*

$$\frac{-1}{\|\Lambda\|_F} \nabla_{xx} (\Lambda \bullet G(u)) \succ 0 \quad \forall \, u \in \partial S, \quad \forall \, 0 \neq \Lambda \succeq 0.$$

Theorem 3.2. *Suppose $S \subseteq int(\mathcal{D})$, $\mathcal{D}$ is compact convex, and there exists $\tilde{x} \in \mathcal{D}$ such that $G(\tilde{x}) \succ 0$. If Assumption 3.1 holds, then $S = L_N$ for all N big enough.*

Proof. For a matrix polynomial $G(x)$ given in (1.2), we define its norm $\|G(x)\|$ as

$$\|G(x)\| = \max_{\alpha \in \mathbb{N}_{\leq 2d}} \frac{\alpha_1! \cdots \alpha_n!}{|\alpha|!} \|G_\alpha\|_2.$$

Since $\mathcal{D}$ is compact, there must exist $\Delta > 0$ such that

$$\frac{1}{\|\Lambda\|_F} \|H(u,x)\| \leq \Delta \quad \forall \Lambda \in \mathbb{R}^{n \times n}, \quad \forall u \in \partial S, \quad \forall x \in \mathcal{D}.$$

Here $H(u,x)$ is defined in (3.5). Assumption 3.1 implies $H(u,x) \succ 0$ for all $u \in \partial S$ and $x \in \mathcal{D}$. This is because otherwise if $H(u,x)$ is not positive definite, we can find $0 \neq v \in \mathbb{R}^n$ such that $v^T H(u,x) v = 0$, i.e.,

$$\int_0^1 \int_0^t v^T \Big(-\nabla_{xx}\big(\Lambda \bullet G(u + s(x-u)\big)\Big) v \, ds \, dt = 0.$$

Since $G(x)$ is matrix concave on the convex domain $\mathcal{D}$, we must have

$$v^T \Big(-\nabla_{xx}\big(\Lambda \bullet G(u + s(x-u))\big)\Big) v = 0 \quad \forall\, 0 \leq s \leq t \leq 1.$$

In particular, we get $v^T \Big(-\nabla_{xx}\big(\Lambda \bullet G(u)\big)\Big) v = 0$, which contradicts Assumption 3.1. Therefore, by the compactness of ∂S and $\mathcal{D}$, there exists $\delta > 0$ such that

$$\frac{1}{\|\Lambda\|_F} H(u,x) \succeq \delta I_n \quad \forall u \in \partial S, \quad \forall x \in \mathcal{D}, \quad \forall\, 0 \neq \Lambda \succeq 0.$$

By Theorem 27 in [6], there exists an integer N^* such that for every $0 \neq \Lambda \succeq 0$ and $u \in \partial S$, there exist sos matrices $F_0(x), F_1(x), \ldots, F_m(x)$ satisfying

$$\frac{1}{\|\Lambda\|_F} H(u,x) = \sum_{k=0}^{m} g_k(x) F_k(x), \tag{3.6}$$

$$\deg(F_k) + 2d_k \leq 2N^*, \; k = 0, , \ldots, m.$$

Now we claim $S = L_{N^*}$. Since $S \subseteq L_{N^*}$, we need to show $L_{N^*} \subseteq S$. Suppose otherwise there exists $\hat{x} \in L_{N^*}/S$. Since $\mathcal{D}$ is compact convex and $G(x)$ is matrix concave on $\mathcal{D}$, S is closed and convex. By Hahn-Banach Theorem, there exist $0 \neq \ell \in \mathbb{R}^n$ and $u \in \partial S$ satisfying

$$\ell^T(x-u) \geq 0 \quad \forall x \in S, \quad \ell^T(\hat{x}-u) < 0.$$

Consider the linear optimization (3.3) with this ℓ. The point $u \in \partial S$ is a minimizer. Since $S \subset int(\mathcal{D})$, the constraint $x \in \mathcal{D}$ is not active in (3.3). The existence of $\tilde{x} \in \mathcal{D}$ with $G(\tilde{x}) \succ 0$ (the Slater's condition holds) implies there exists $\Lambda \succeq 0$ satisfying (3.4). From (3.5) and (3.6), we know there are SOS polynomials $p_0(x)$, $p_1(x)$, $\ldots$, $p_m(x)$ such that

$$\ell^T(x-u) = \Lambda \bullet G(x) + \sum_{k=0}^{m} p_k(x) g_k(x),$$

and they satisfy

$$\deg(p_k) + 2d_k \leq 2N^*, \quad k = 0, 1, \ldots, m.$$

So there are symmetric matrices $W_0, W_1, \ldots, W_m \succeq 0$ such that

$$\ell^T(x-u) = \Lambda \bullet G(x) + \sum_{k=0}^{m} g_k(x)[x]_{N^*-d_k}^T W_k [x]_{N^*-d_k}.$$

By definition of matrices $B_{k,\beta}^{(N^*)}$ in (3.1), it holds the identity

$$\ell^T(x-u) = \Lambda \bullet G(x) + \sum_{k=0}^{m} W_k \bullet \left(\sum_{\beta \in \mathbb{N}_{\leq 2d}} x^\beta B_{k,\beta}^{(N^*)} \right).$$

By the choice of $\hat{x}$, there exists $\hat{y}$ such that $\hat{x} = (\hat{y}_{e_1}, \ldots, \hat{y}_{e_n})$, $\mathcal{G}(\hat{y}) \succeq 0$, and every $B_k^{N^*}(\hat{y}) \succeq 0$. So in the above, if every monomial $\hat{x}^\alpha$ is replaced by $\hat{y}_\alpha$, then

$$\ell^T(\hat{x}-u) = \Lambda \bullet \mathcal{G}(\hat{y}) + \sum_{k=0}^{m} W_k \bullet B_k^{N^*}(\hat{y}) \geq 0,$$

which contradicts $\ell^T(\hat{x}-u) < 0$. So we must have $S = L_{N^*}$.

For every $N \geq N^*$, the relation $S \subseteq L_N \subseteq L_{N^*}$ implies $S = L_N$. So the theorem is proven. □

Assumption 3.1 requires us to check $-\nabla_{xx}(\Lambda \bullet G(u)) \succ 0$ for every nonzero $\Lambda \succeq 0$ and $u \in \partial S$, which is sometimes very inconvenient. However, Assumption 3.1 is true if $G(x)$ is *strictly matrix concave* on $\mathcal{D}$, that is, for every $0 \neq \xi \in \mathbb{R}^m$, the Hessian $-\nabla_{xx}(\xi^T G(x)\xi) \succ 0$ for all $x \in \mathcal{D}$. So the following is a consequence of Theorem 3.2.

Corollary 3.3. *Suppose $S \subseteq int(\mathcal{D})$, $\mathcal{D}$ is compact convex, and there exists $\tilde{x} \in \mathcal{D}$ such that $G(\tilde{x}) \succ 0$. If $G(x)$ is strictly matrix concave on $\mathcal{D}$, then $S = L_N$ for all N big enough.*

4 Rational matrix inequality

In this section, assume $S = \{x \in \mathcal{D} : G(x) \succeq 0\}$ is defined by a matrix rational function $G(x)$, i.e., every entry of $G(x)$ is a rational function. Suppose $G(x)$ is matrix-concave on $\mathcal{D}$. As before, the domain $\mathcal{D} = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}$ is still defined by multivariate polynomials. When $G(x)$ is a scalar rational function, the author in [19] studied the SDP representation of S. This section discusses the more general case of $G(x)$ being a matrix. We will first construct an SDP relaxation for S, and then prove it represents S when $G(x)$ satisfies a certain condition.

Suppose the matrix rational function $G(x)$ is given as

$$G(x) = \frac{1}{den(G)} \sum_{\alpha \in \mathbb{N}^n : |\alpha| \leq \deg(G)} x^\alpha F_\alpha, \tag{4.1}$$

where $F_\alpha \in \mathbb{R}^{m \times m}$ are symmetric matrices, $den(G)$ is the common denominator of $G(x)$, and $\deg(G)$ is the degree of $G(x)$. Let $p(x), q(x)$ be two given polynomials which are positive in

$int(\mathcal{D})$. We say $G(x)$ is *q-module matrix concave* over $\mathcal{D}$ with respect to (p,q) if for every $\xi \in \mathbb{R}^m$, there exist SOS polynomials $\sigma_{i,j}(x,u)$ such that

$$\begin{gathered} p(x)q(u) \cdot \left(\xi^T G(u) \xi + (\nabla_x \xi^T G(x) \xi)^T \Big|_{x=u} (x-u) - \xi^T G(x) \xi \right) = \\ \sum_{i=0}^{m} g_i(x) \left(\sum_{j=0}^{m} g_j(u) \sigma_{ij}(x,u) \right) \end{gathered} \tag{4.2}$$

is an identity in (x,u). In the above $g_0(x)=1$. If (p,q) can be chosen as $(den(G), den(G)^2)$, then we just say $G(x)$ is q-module matrix concave over $\mathcal{D}$. The condition (4.2) is based on Putinar's Positivstellensatz [22]. Clearly, if $G(x)$ is q-module matrix concave over $\mathcal{D}$ with respect to (p,q), then it must also be matrix concave over $\mathcal{D}$.

Now we turn to the construction of lifted LMI for S. Assume $G(x)$ is q-module matrix concave over $\mathcal{D}$ with respect to (p,q) and (4.2) holds. Set integer

$$d = \max\left\{ \max_{0 \le i,j \le m} \quad \lceil \tfrac{1}{2} \deg_x(g_i \sigma_{ij}) \rceil, \tfrac{1}{2}\deg(G) \right\}. \tag{4.3}$$

For $i = 0, \ldots, m$, define matrices $P_\alpha^{(i)}, Q_\alpha^{(i)}$ such that

$$\tfrac{g_i(x)}{p(x)} [x]_{d-d_i} [x]_{d-d_i}^T \quad = \sum_{\alpha \in \mathbb{N}^n : |\alpha| + |LE(p)| \le 2d} Q_\alpha^{(i)} x^\alpha + \sum_{\beta \in \mathbb{N}_{\le 2d} : \beta < LE(p)} P_\alpha^{(i)} \tfrac{x^\beta}{p(x)}. \tag{4.4}$$

Here $LE(p)$ denotes the exponent of the leading monomial of $p(x)$ in the alphabetical ordering $(x_1 > x_2 > \cdots > x_n)$. Let y be a vector indexed by $\alpha \in \mathbb{N}^n$ with $|\alpha| + |LE(p)| \le 2d$, and z be a vector indexed by $\beta \in \mathbb{N}_{\le 2d}$ with $\beta < LE(p)$. Define linear matrix pencils

$$Q_i(y,z) = \sum_{\alpha \in \mathbb{N}^n : |\alpha| + |LE(p)| \le 2d} Q_\alpha^{(i)} y_\alpha + \sum_{\beta \in \mathbb{N}_{\le 2d} : \beta < LE(p)} P_\alpha^{(i)} z_\beta, \quad i = 0, 1, \ldots, m. \tag{4.5}$$

Here $P_\alpha^{(i)}, Q_\alpha^{(i)}$ are defined in (4.4). Suppose $G(x)$ is given as

$$G(x) = \sum_{\alpha \in \mathbb{N}^n : |\alpha| + |LE(p)| \le 2d} F_\alpha^{(1)} x^\alpha + \sum_{\beta \in \mathbb{N}_{\le 2d} : \beta < LE(p)} F_\beta^{(2)} \frac{x^\beta}{p(x)},$$

then define linear matrix pencil

$$F(y,z) = \sum_{\alpha \in \mathbb{N}^n : |\alpha| + |LE(p)| \le 2d} F_\alpha^{(1)} y_\alpha + \sum_{\beta \in \mathbb{N}_{\le 2d} : \beta < LE(p)} F_\beta^{(2)} z_\beta.$$

Clearly, S can be equivalently described as

$$S = \left\{ (y_{e_1}, \ldots, y_{e_n}) \in \mathbb{R}^n : \begin{array}{l} \exists\, x \in \mathbb{R}^n,\, y_\alpha = x^\alpha, z_\beta = \frac{x^\beta}{p(x)} \quad \forall\, \alpha, \beta \\ F(y,z) \succeq 0,\, Q_i(y,z) \succeq 0,\, i = 0, \ldots, m \end{array} \right\}.$$

If we remove $y_\alpha = x^\alpha, z_\beta = \frac{x^\beta}{p(x)}$ in the above, then S is a subset of

$$L_{qmod} = \left\{ x \in \mathbb{R}^n : \begin{array}{l} \exists\, y, z,\, y_0 = 1, x_1 = y_{e_1}, \ldots, x_n = y_{e_n}, \\ F(y,z) \succeq 0,\, Q_i(y,z) \succeq 0,\, i = 0, \ldots, m \end{array} \right\}. \tag{4.6}$$

So $S \subseteq L_{qmod}$. We are interested in conditions making $S = L_{qmod}$.

Lemma 4.1. *Assume $S \subset int(\mathcal{D})$ and there exists $\tilde{x} \in \mathcal{D}$ such that $G(\tilde{x}) \succ 0$. Suppose the matrix rational function $G(x)$ is q-module matrix concave over $\mathcal{D}$ with respect to (p, q). If $v \in \partial S$, $q(v) > 0$, $den(G)(v) > 0$, and $a^T(x - v) \geq 0$ for all $x \in S$, then*

$$p(x) \cdot (a^T(x - v) - \Lambda \bullet G(x)) = \sum_{i=0}^{m} g_i(x)\sigma_i(x)$$

for some symmetric matrix $\Lambda \succeq 0$ and sos polynomials $\sigma_i(x)$ with $\deg(g_i\sigma_i) \leq 2d$.

Proof. Consider the linear optimization

$$\min_{x\in\mathcal{D}} \quad a^T x \quad \text{subject to} \quad G(x) \succeq 0.$$

The point $v \in \partial S$ is an optimizer. Since $den(G)(v) > 0$, $G(x)$ is differentiable at v. Since $S \subset int(\mathcal{D})$, the constraint $x \in \mathcal{D}$ is not active at v. Because of the existence of $\tilde{x} \in \mathcal{D}$ with $G(\tilde{x}) \succ 0$ (the Slater's condition holds) and $G(x)$ being matrix concave on $\mathcal{D}$, there exists a Lagrange multiplier $\Lambda \succeq 0$ such that

$$a = \nabla_x(\Lambda \bullet G(v)), \quad \Lambda \bullet G(v) = 0.$$

Hence, we get the identity

$$a^T(x - v) - \Lambda \bullet G(x) = \Lambda \bullet G(v) + \nabla_x(\Lambda \bullet G(v))^T(x - v) - \Lambda \bullet G(x).$$

Since $\Lambda \succeq 0$, we have decomposition $\Lambda = \sum_{i=1}^{K} \lambda^{(i)}(\lambda^{(i)})^T$. Then it holds

$$a^T(x - v) - \Lambda \bullet G(x) =$$
$$\sum_{i=1}^{K} \Big\{(\lambda^{(i)})^T G(v)\lambda^{(i)} + \nabla_x((\lambda^{(i)})^T G(v)\lambda^{(i)})^T(x - v) - (\lambda^{(i)})^T G(x)\lambda^{(i)}\Big\}.$$

So the lemma readily follows the q-module matrix concavity of $G(x)$. □

For a function $f(x)$, denote by $\mathcal{Z}(f)$ its real zero set, i.e., $\mathcal{Z}(f) = \{x \in \mathbb{R}^n : f(x) = 0\}$.

Theorem 4.2. *Assume S is closed and convex, $S \subset int(\mathcal{D})$, $G(\tilde{x}) \succ 0$ for some $\tilde{x} \in \mathcal{D}$, and*

$$\dim(\mathcal{Z}\big(\det(G)) \cap \mathcal{Z}(den(G)) \cap \partial S\big) < n - 1, \dim\big(\mathcal{Z}(q) \cap \partial S\big) < n - 1.$$

If $G(x)$ is q-module matrix concave over $\mathcal{D}$ with respect to (p, q), then $S = L_{qmod}$.

Proof. Since $S \subseteq L_{qmod}$, it is sufficient for us to prove the reverse containment. For a contradiction, suppose there exists $\hat{x} \in L_{qmod}/S$ and $(\hat{y}, \hat{z})$ such that

$$\hat{x} = (\hat{y}_{e_1}, \ldots, \hat{y}_{e_n}), \quad F(\hat{y}, \hat{z}) \succeq 0, \quad Q_i(\hat{y}, \hat{z}) \succeq 0, \, i = 0, \ldots, m.$$

Since S is convex and closed, by Hahn-Banach Theorem, there exists a supporting hyperplane $\{a^T x = b\}$ of S such that $a^T x \geq b$ for all $x \in S$ and $a^T \hat{x} < b$. Let $v \in \partial S$ be a minimizer of $a^T x$ on S. Since $\dim\big(\det(G) \cap \mathcal{Z}(p) \cap \partial S\big) < n - 1$ and $\dim\big(\mathcal{Z}(q) \cap \partial S\big) < n - 1$, by continuity, the supporting hyperplane $\{a^T x = b\}$ can be chosen to satisfy $deg(G)(v) > 0$, and $q(v) > 0$. By Lemma 4.1, we have

$$a^T(x - v) = \Lambda \bullet G(x) + \sum_{i=0}^{m} \frac{g_i(x)}{p(x)} \sigma_i(x) \tag{4.7}$$

for some sos polynomials $\sigma_i(x)$ such that every $\deg(g_i \sigma_i) \le 2d$. If we write $\sigma_i(x)$ as

$$\sigma_i(x) = [x]_{d-d_i}^T W_i [x]_{d-d_i}$$

for symmetric $W_i \succeq 0 \, (i = 0, 1, \ldots, m)$, then identity (4.7) becomes

$$\begin{aligned} a^T(x - v) &= \Lambda \bullet G(x) + \sum_{i=0}^{m} \left(\frac{g_i(x)}{p(x)} [x]_{d-d_i} [x]_{d-d_i}^T \right) \bullet W_i \\ &= \Lambda \bullet G(x) + \sum_{i=0}^{m} \left(\sum_{\alpha \in \mathbb{N}^n : |\alpha| + |LE(p)| \le 2d} Q_\alpha^{(i)} x^\alpha + \sum_{\beta \in \mathbb{N}^n : \beta < LE(p)} P_\alpha^{(i)} \frac{x^\beta}{p(x)} \right) \bullet W_i. \end{aligned}$$

In the above identity, if we replace every x^α by $\hat{y}_\alpha$ and $\frac{x^\beta}{p(x)}$ by $\hat{z}_\beta$, then

$$a^T \hat{x} - b = \Lambda \bullet F(\hat{y}, \hat{z}) + \sum_{i=0}^{m} Q_i(\hat{y}, \hat{z}) \bullet W_i \ge 0,$$

because all $\Lambda, F(\hat{y}, \hat{z}), Q_i(\hat{y}, \hat{z}), W_i \succeq 0$. This contradicts $a^T \hat{x} < b$. So $S = L_{qmod}$. □

The condition of q-module matrix concavity requires checking (4.2) for every $\xi \in \mathbb{R}^n$. In many situations this is almost impossible. However, if we consider ξ as an indeterminant, then one sufficient condition guaranteeing (4.2) is

$$\begin{aligned} p(x)q(u) \cdot \left(\xi^T G(u) \xi + (\nabla_x \xi^T G(x) \xi)^T \Big|_{x=u} (x - u) - \xi^T G(x) \xi \right) = \\ \sum_{i=0}^{m} g_i(x) \left(\sum_{j=0}^{m} g_j(u) \sigma_{ij}(x, u, \xi) \right), \end{aligned} \tag{4.8}$$

where every $\sigma_{ij}(x, u, \xi)$ is now an SOS polynomial in (x, u, ξ). If $G(x)$ satisfies condition (4.8), we say $G(x)$ is *uniformly q-module matrix concave* over $\mathcal{D}$ with respect to (p, q). If in addition $p = den(G)(x), q = deg(G)^2(u)$, then we just say $G(x)$ is *uniformly q-module matrix concave* over $\mathcal{D}$. So the following corollary follows Theorem 4.2.

Corollary 4.3. *Assume S is closed and convex, $S \subset int(\mathcal{D})$, $G(\tilde{x}) \succ 0$ for some $\tilde{x} \in \mathcal{D}$, and*

$$\dim(\mathcal{Z}(\det(G)) \cap \mathcal{Z}(den(G)) \cap \partial S) < n - 1, \, \dim(\mathcal{Z}(q) \cap \partial S) < n - 1.$$

If $G(x)$ is uniformly q-module matrix concave over $\mathcal{D}$ with respect to (p, q), then $S = L_{qmod}$.

Now we give some examples on how to apply Theorem 4.2 and Corollary 4.3 to get an SDP representation for S when $G(x)$ is a matrix rational function.

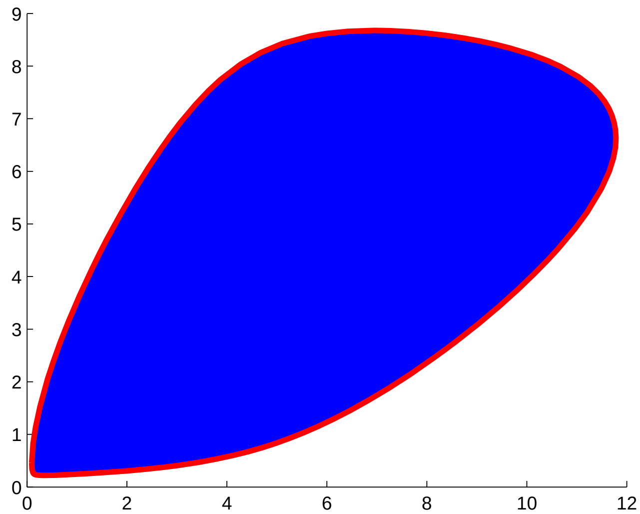

Figure 3: The shaded area is the convex set in Example 4.4, and the thick curve is its boundary.

Example 4.4. Consider the set $\{x \in \mathbb{R}^2_+ : G(x) \succeq 0\}$ where

$$G(x) = \begin{bmatrix} 7 - x_1 + 2x_2 & 5 \\ 5 & 11 - x_2 \end{bmatrix} - \frac{1}{x_1 x_2} \begin{bmatrix} x_1 + x_2^3 & x_2^2 \\ x_2^2 & x_2 \end{bmatrix}.$$

Its domain $\mathcal{D}$ is the nonnegative orthant. A plot of this set is in Figure 3. The above $G(x)$ is uniformly q-module matrix concave over $\mathbb{R}^2_+$, because

$$x_1 x_2 u_1^2 u_1^2 \cdot \left(\xi^T G(u)\xi + (\nabla_x \xi^T G(x)\xi)^T \Big|_{x=u} (x-u) - \xi^T G(x)\xi \right) = \\ x_2 u_2^2 (u_1\xi_2 - x_1\xi_2 + u_1 x_2 \xi_1 - u_2 x_1 \xi_1)^2 + x_1 u_1^2 \xi_1^2 (u_2 - x_2)^2.$$

Hence, this set is convex, and by Corollary 4.3 it has the following lifted LMI

$$\begin{bmatrix} 7 - x_1 + 2x_2 & 5 \\ 5 & 11 - x_2 \end{bmatrix} - \begin{bmatrix} z_{10} + z_{03} & z_{02} \\ z_{02} & z_{01} \end{bmatrix} \succeq 0,$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & x_1 & x_2 \\ 0 & 1 & 0 & x_1 & x_2 & 0 \\ 0 & 0 & x_1 & 0 & y_{20} & y_{11} \\ 1 & x_1 & x_2 & y_{20} & y_{11} & y_{02} \\ 0 & x_2 & 0 & y_{11} & y_{02} & 0 \end{bmatrix} + \begin{bmatrix} z_{00} & z_{10} & z_{01} & z_{20} & 0 & z_{02} \\ z_{10} & z_{20} & 0 & z_{30} & 0 & 0 \\ z_{01} & 0 & z_{02} & 0 & 0 & z_{03} \\ z_{20} & z_{30} & 0 & z_{40} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ z_{02} & 0 & z_{03} & 0 & 0 & z_{04} \end{bmatrix} \succeq 0,$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & x_1 \\ 1 & x_1 & x_2 \end{bmatrix} + \begin{bmatrix} z_{10} & z_{20} & 0 \\ z_{20} & z_{30} & 0 \\ 0 & 0 & 0 \end{bmatrix} \succeq 0, \begin{bmatrix} 0 & 1 & 0 \\ 1 & x_1 & x_2 \\ 0 & x_2 & 0 \end{bmatrix} + \begin{bmatrix} z_{01} & 0 & z_{02} \\ 0 & 0 & 0 \\ z_{02} & 0 & z_{03} \end{bmatrix} \succeq 0.$$

The plot of x in the above coincides with the shaded area of Figure 3, which confirms this lifted LMI is valid.

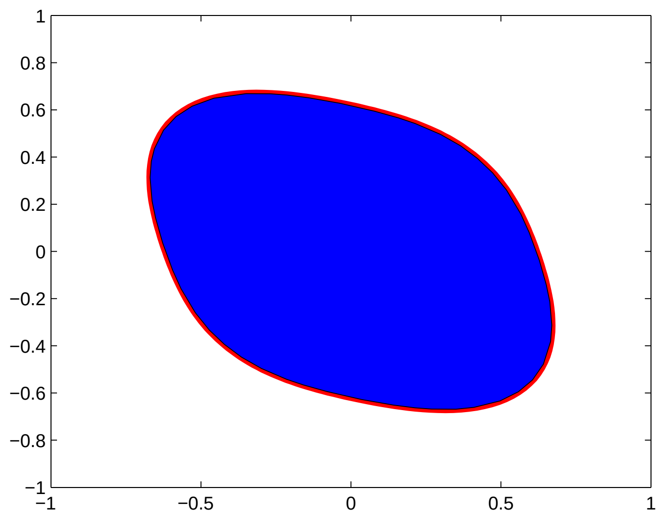

Figure 4: The shaded area is the convex set in Example 4.5, and the thick curve is its boundary.

Example 4.5. Consider the set $\{x \in \mathbb{R}^2 : G(x) \succeq 0\}$ where

$$G(x) = \begin{bmatrix} 1-2x_1^2-2x_1x_2-x_2^2 & x_1^2 \\ x_1^2 & 1-x_1^2 \end{bmatrix} + \frac{x_2^4}{x_1^2+x_2^2}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}^T.$$

A plot of this set is in Figure 4. Its domain is the whole plane $\mathbb{R}^2$, and the above $G(x)$ is uniformly q-module matrix concave over $\mathbb{R}^2$, because

$$\|x\|_2^2 \cdot \|u\|_2^4 \cdot \left(\xi^T G(u)\xi + (\nabla_x \xi^T G(x)\xi)^T\Big|_{x=u}(x-u) - \xi^T G(x)\xi\right) = \left(\sum_{i=1}^{9} f_i^2\right) \cdot (\xi_1-\xi_2)^2 + \tfrac{1}{2}\|x\|_2^2 \cdot \|u\|_2^4 \cdot \|x-u\|_2^2 \cdot \xi_1^2$$

where the polynomials f_i are given as below

$$\begin{aligned}
f_1 &= -u_1u_2x_2^2 - u_1u_2x_1^2 + u_1u_2^2x_2 + u_1^2u_2x_1, & f_6 &= \tfrac{1}{\sqrt{2}}(-u_2^2x_2^2 + u_2^3x_2 - u_1^2x_1^2 + u_1^3x_1),\\
f_2 &= -u_1u_2x_2^2 + u_1u_2x_1^2 + u_1u_2^2x_2 - u_1^2u_2x_1, & f_7 &= -2u_1u_2x_1x_2 + u_1u_2^2x_1 + u_1^2u_2x_2,\\
f_3 &= \tfrac{1}{\sqrt{2}}(-u_2^2x_1x_2 + u_2^3x_1 - u_1^2x_1x_2 + u_1^3x_2), & f_8 &= u_2^2x_1^2 - u_1^2x_2^2,\\
f_4 &= \tfrac{1}{\sqrt{2}}(u_2^2x_1x_2 - u_2^3x_1 - u_1^2x_1x_2 + u_1^3x_2), & f_9 &= -u_1u_2^2x_1 + u_1^2u_2x_2.\\
f_5 &= \tfrac{1}{\sqrt{2}}(u_2^2x_2^2 - u_2^3x_2 - u_1^2x_1^2 + u_1^3x_1), & &
\end{aligned}$$

So this set is convex, and by Corollary 4.3 a lifted LMI for it is

$$\begin{bmatrix} 1-2y_{20}-2y_{11}-y_{02}-z_{04} & y_{20}+z_{04} \\ y_{20}+z_{04} & 1-y_{20}-z_{04} \end{bmatrix} \succeq 0,$$

$$
\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & x_1 & x_2 & 0 \\ 0 & 0 & 0 & x_2 & 0 & 0 \\ 1 & x_1 & x_2 & y_{20}-y_{02} & y_{11} & y_{02} \\ 0 & x_2 & 0 & y_{11} & y_{02} & 0 \\ 0 & 0 & 0 & y_{02} & 0 & 0 \end{bmatrix} + \begin{bmatrix} z_{00} & z_{10} & z_{01} & -z_{02} & z_{11} & z_{02} \\ z_{10} & -z_{02} & z_{11} & -z_{12} & -z_{03} & z_{12} \\ z_{01} & z_{11} & z_{02} & -z_{03} & z_{12} & z_{03} \\ -z_{02} & -z_{12} & -z_{03} & z_{04} & -z_{13} & -z_{04} \\ z_{11} & -z_{03} & z_{12} & -z_{13} & -z_{04} & z_{13} \\ z_{02} & z_{12} & z_{03} & -z_{04} & z_{13} & z_{04} \end{bmatrix} \succeq 0.
$$

The plot of x in the above lifted LMI coincides with the shaded area of Figure 4, which confirms it is valid.

5 Conclusions

This paper gave explicit constructions of SDP representations for the set $S = \{x \in \mathcal{D} : G(x) \succeq 0\}$ when $G(x)$ is a matrix polynomial or rational function, and proved sufficient conditions validating them. These conditions are generally based on the matrix concavity of $G(x)$. We would like to point out that $G(x)$ might not be matrix concave when S is convex. For instance, for the quadratic polynomial matrix

$$
Q(x) = \begin{bmatrix} x_1x_2+2 & x_1x_2 & 0 \\ x_1x_2 & x_1x_2-1 & 0 \\ 0 & 0 & x_1+x_2 \end{bmatrix},
$$

the matrix inequality $Q(x) \succeq 0$ defines the convex set $\{x \in \mathbb{R}^2_+ : x_1x_2 \geq 2\}$, but $Q(x)$ is not matrix concave on $\mathbb{R}^2_+$. In such cases, the lifted LMIs constructed in this paper might no longer represent S. It is an important future work to find SDP representations for S when $G(x)$ is not matrix concave. On the other hand, if S is compact convex and its boundary is nonsingular and positively curved, it is shown in [7] that S has a lifted LMI. But it is not clear how to find an efficient one. This is another interesting future work. More recent results on semidefinite representability of convex semialgebraic sets can be found in [1, 10, 11, 6, 7, 8, 12, 13, 14, 19].

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