4. The point of intersection satisfies the system of two linear equations:

$$x_1 - 5x_2 = 1 \quad\quad \begin{bmatrix} 1 & -5 & 1 \\ 3 & -7 & 5 \end{bmatrix}$$
$$3x_1 - 7x_2 = 5$$

Replace R2 by R2 + (–3)R1 and obtain:

$$\begin{aligned} x_1 - 5x_2 &= 1 \\ 8x_2 &= 2 \end{aligned} \quad\quad \begin{bmatrix} 1 & -5 & 1 \\ 0 & 8 & 2 \end{bmatrix}$$

Scale R2 by 1/8:

$$\begin{aligned} x_1 - 5x_2 &= 1 \\ x_2 &= 1/4 \end{aligned} \quad\quad \begin{bmatrix} 1 & -5 & 1 \\ 0 & 1 & 1/4 \end{bmatrix}$$

Replace R1 by R1 + (5)R2:

$$\begin{aligned} x_1 \quad &= 9/4 \\ x_2 &= 1/4 \end{aligned} \quad\quad \begin{bmatrix} 1 & 0 & 9/4 \\ 0 & 1 & 1/4 \end{bmatrix}$$

The point of intersection is $(x_1, x_2) = (9/4, 1/4)$.

6. One more step will put the system in triangular form. Replace R4 by its sum with –3 times R3, which produces $\begin{bmatrix} 1 & -6 & 4 & 0 & -1 \\ 0 & 2 & -7 & 0 & 4 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & -5 & 15 \end{bmatrix}$. After that, the next step is to scale the fourth row by –1/5.

12. Replace R2 by R2 + (–3)R1 and replace R3 by R3 + (4)R1. Finally, replace R3 by R3 + (3)R2.

$$\begin{bmatrix} 1 & -3 & 4 & -4 \\ 3 & -7 & 7 & -8 \\ -4 & 6 & -1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & -6 & 15 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

The system is inconsistent, because the last row would require that 0 = 3 if there were a solution. The solution set is empty.

16. First replace R4 by R4 + (2)R1 and replace R4 by R4 + (–3/2)R2. (One could also scale R2 before adding to R4, but the arithmetic is rather easy keeping R2 unchanged.) Finally, replace R4 by R4 + R3.

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ -2 & 3 & 2 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 3 & 2 & -3 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The system is now in triangular form and has a solution. The next section discusses how to continue with this type of system.

20. $\begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & h & -3 \\ 0 & 4+2h & 0 \end{bmatrix}$. Write c for $4 + 2h$. Then the second equation $cx_2 = 0$ has a solution for every value of c. So the system is consistent for all h.

30. Multiply R2 by –1/2; multiply R2 by –2.

2. Reduced echelon form: a. Echelon form: b and d. Not echelon: c.

4.
$$\begin{bmatrix}1&3&5&7\\3&5&7&9\\5&7&9&1\end{bmatrix}\sim\begin{bmatrix}1&3&5&7\\0&-4&-8&-12\\0&-8&-16&-34\end{bmatrix}\sim\begin{bmatrix}1&3&5&7\\0&1&2&3\\0&-8&-16&-34\end{bmatrix}\sim\begin{bmatrix}1&3&5&7\\0&1&2&3\\0&0&0&-10\end{bmatrix}$$

$$\sim\begin{bmatrix}1&3&5&7\\0&1&2&3\\0&0&0&1\end{bmatrix}\sim\begin{bmatrix}1&3&5&0\\0&1&2&0\\0&0&0&1\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&0&-1&0\\0&\textcircled{1}&2&0\\0&0&0&\textcircled{1}\end{bmatrix}.\quad \text{Pivot cols 1, 2, and 4}\quad \begin{bmatrix}\textcircled{1}&3&5&7\\3&\textcircled{5}&7&9\\5&7&9&\textcircled{1}\end{bmatrix}$$

8.
$$\begin{bmatrix}1&4&0&7\\2&7&0&10\end{bmatrix}\sim\begin{bmatrix}1&4&0&7\\0&-1&0&-4\end{bmatrix}\sim\begin{bmatrix}1&4&0&7\\0&1&0&4\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&0&0&-9\\0&\textcircled{1}&0&4\end{bmatrix}$$

Corresponding system of equations:
$$\begin{aligned}\textcircled{x_1}\quad &= -9\\ \textcircled{x_2}\quad &= 4\end{aligned}$$

The basic variables (corresponding to the pivot positions) are x_1 and x_2. The remaining variable x_3 is free. Solve for the basic variables in terms of the free variable. In this particular problem, the basic variables do not depend on the value of the free variable.

General solution: $\begin{cases}x_1 = -9\\ x_2 = 4\\ x_3 \text{ is free}\end{cases}$

Note: A common error in Exercise 8 is to assume that x_3 is zero. To avoid this, identify the basic variables first. Any remaining variables are *free*. (This type of computation will arise in Chapter 5.)

10.
$$\begin{bmatrix}1&-2&-1&3\\3&-6&-2&2\end{bmatrix}\sim\begin{bmatrix}1&-2&-1&3\\0&0&1&-7\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&-2&0&-4\\0&0&\textcircled{1}&-7\end{bmatrix}$$

Corresponding system:
$$\begin{aligned}\textcircled{x_1} - 2x_2 \quad &= -4\\ \textcircled{x_3} &= -7\end{aligned}$$

Basic variables: x_1, x_3; free variable: x_2. General solution: $\begin{cases}x_1 = -4 + 2x_2\\ x_2 \text{ is free}\\ x_3 = -7\end{cases}$

12.
$$\begin{bmatrix}1&-7&0&6&5\\0&0&1&-2&-3\\-1&7&-4&2&7\end{bmatrix}\sim\begin{bmatrix}1&-7&0&6&5\\0&0&1&-2&-3\\0&0&-4&8&12\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&-7&0&6&5\\0&0&\textcircled{1}&-2&-3\\0&0&0&0&0\end{bmatrix}$$

Corresponding system:
$$\begin{aligned}\textcircled{x_1} - 7x_2 \quad + 6x_4 &= 5\\ \textcircled{x_3} - 2x_4 &= -3\\ 0 &= 0\end{aligned}$$

Basic variables: x_1 and x_3; free variables: x_2, x_4. General solution: $\begin{cases}x_1 = 5 + 7x_2 - 6x_4\\ x_2 \text{ is free}\\ x_3 = -3 + 2x_4\\ x_4 \text{ is free}\end{cases}$

14. $\begin{bmatrix} 1 & 2 & -5 & -6 & 0 & -5 \\ 0 & 1 & -6 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 7 & 0 & 0 & -9 \\ 0 & \textcircled{1} & -6 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Corresponding system:

$$\begin{array}{rcrcrcrcl} \textcircled{x_1} & & + & 7x_3 & & & & = & -9 \\ & \textcircled{x_2} & - & 6x_3 & - & 3x_4 & & = & 2 \\ & & & & & & \textcircled{x_5} & = & 0 \\ & & & & & & 0 & = & 0 \end{array}$$

Basic variables: x_1, x_2, x_5; free variables: x_3, x_4. General solution: $\begin{cases} x_1 = -9 - 7x_3 \\ x_2 = 2 + 6x_3 + 3x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \\ x_5 = 0 \end{cases}$

20. $\begin{bmatrix} 1 & 3 & 2 \\ 3 & h & k \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 3 & 2 \\ 0 & h-9 & k-6 \end{bmatrix}$

a. When $h = 9$ and $k \neq 6$, the system is inconsistent, because the augmented column is a pivot column.

b. When $h \neq 9$, the system is consistent and has a unique solution. There are no free variables.

c. When $h = 9$ and $k = 6$, the system is consistent and has many solutions.

24. The system is inconsistent because the pivot in column 5 means that there is a row of the form $[0 \quad 0 \quad 0 \quad 0 \quad 1]$. Since the matrix is the *augmented* matrix for a system, Theorem 2 shows that the system has no solution.