1.3

6. $x_1\begin{bmatrix}-2\\3\end{bmatrix}+x_2\begin{bmatrix}8\\5\end{bmatrix}+x_3\begin{bmatrix}1\\-6\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}-2x_1\\3x_1\end{bmatrix}+\begin{bmatrix}8x_2\\5x_2\end{bmatrix}+\begin{bmatrix}x_3\\-6x_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}-2x_1+8x_2+x_3\\3x_1+5x_2-6x_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$$\begin{aligned}-2x_2 &+ 8x_2 + x_3 = 0\\ 3x_1 &+ 5x_2 - 6x_3 = 0\end{aligned}$$

Usually the intermediate steps are not displayed.

12. The equation

$$x_1\begin{bmatrix}1\\-2\\2\end{bmatrix}+x_2\begin{bmatrix}0\\5\\5\end{bmatrix}+x_3\begin{bmatrix}2\\0\\8\end{bmatrix}=\begin{bmatrix}-5\\11\\-7\end{bmatrix} \qquad (*)$$

$$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow$$

$$\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$M=\begin{bmatrix}1&0&2&-5\\-2&5&0&11\\2&5&8&-7\end{bmatrix}$$

Row reduce M until the pivot positions are visible:

$$M\sim\begin{bmatrix}1&0&2&-5\\0&5&4&1\\0&5&4&3\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&0&2&-5\\0&\textcircled{5}&4&1\\0&0&0&\textcircled{2}\end{bmatrix}$$

The linear system corresponding to M has *no* solution, so the vector equation (*) has no solution, and therefore $\mathbf{b}$ is *not* a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

16. Some likely choices are $0\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \mathbf{0}$, and

$$1\cdot\mathbf{v}_1+0\cdot\mathbf{v}_2=\begin{bmatrix}3\\0\\2\end{bmatrix},\quad 0\cdot\mathbf{v}_1+1\cdot\mathbf{v}_2=\begin{bmatrix}-2\\0\\3\end{bmatrix}$$

$$1\cdot\mathbf{v}_1+1\cdot\mathbf{v}_2=\begin{bmatrix}1\\0\\5\end{bmatrix},\quad 1\cdot\mathbf{v}_1-1\cdot\mathbf{v}_2=\begin{bmatrix}5\\0\\-1\end{bmatrix}$$

18. $[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{y}]=\begin{bmatrix}1&-3&h\\0&1&-5\\-2&8&-3\end{bmatrix}\sim\begin{bmatrix}1&-3&h\\0&1&-5\\0&2&-3+2h\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&-3&h\\0&\textcircled{1}&-5\\0&0&7+2h\end{bmatrix}$. The vector $\mathbf{y}$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ when $7 + 2h$ is zero, that is, when $h = -7/2$.

22. Construct any 3×4 matrix in echelon form that corresponds to an inconsistent system. Perform sufficient row operations on the matrix to eliminate all zero entries in the first three columns.

26. a. $[\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{a}_3\ \ \mathbf{b}]=\begin{bmatrix}2&0&6&10\\-1&8&5&3\\1&-2&1&3\end{bmatrix}\sim\begin{bmatrix}1&0&3&5\\-1&8&5&3\\1&-2&1&3\end{bmatrix}\sim\begin{bmatrix}1&0&3&5\\0&8&8&8\\0&-2&-2&-2\end{bmatrix}\sim\begin{bmatrix}1&0&3&5\\0&8&8&8\\0&0&0&0\end{bmatrix}$

Yes, $\mathbf{b}$ is a linear combination of the columns of A, that is, $\mathbf{b}$ is in W.

b. The third column of A is in W because $\mathbf{a}_3 = 0\cdot\mathbf{a}_1 + 0\cdot\mathbf{a}_2 + 1\cdot\mathbf{a}_3$.

1.4

4. $A\mathbf{x} = \begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1\cdot\begin{bmatrix} 8 \\ 5 \end{bmatrix} + 1\cdot\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 1\cdot\begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8+3-4 \\ 5+1+2 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$, and

$$A\mathbf{x} = \begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8\cdot 1+3\cdot 1+(-4)\cdot 1 \\ 5\cdot 1+1\cdot 1+2\cdot 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

6. On the left side of the matrix equation, use the entries in the vector **x** as the weights in a linear combination of the columns of the matrix A:

$$-2\cdot\begin{bmatrix} 7 \\ 2 \\ 9 \\ -3 \end{bmatrix} - 5\cdot\begin{bmatrix} -3 \\ 1 \\ -6 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ 12 \\ -4 \end{bmatrix}$$

10. The system has the same solution set as the vector equation

$$x_1\begin{bmatrix} 8 \\ 5 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

and this equation has the same solution set as the matrix equation

$$\begin{bmatrix} 8 & -1 \\ 5 & 4 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

13. The vector **u** is in the plane spanned by the columns of A if and only if **u** is a linear combination of the columns of A. This happens if and only if the equation $A\mathbf{x} = \mathbf{u}$ has a solution. (See the box preceding Example 3 in Section 1.4.) To study this equation, reduce the augmented matrix $[A \;\; \mathbf{u}]$

$$\begin{bmatrix} 3 & -5 & 0 \\ -2 & 6 & 4 \\ 1 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 \\ -2 & 6 & 4 \\ 3 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 8 & 12 \\ 0 & -8 & -12 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 1 & 4 \\ 0 & \textcircled{8} & 12 \\ 0 & 0 & 0 \end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{u}$ has a solution, so **u** is in the plane spanned by the columns of A.

For your information: The unique solution of $A\mathbf{x} = \mathbf{u}$ is (5/2, 3/2).

16. Row reduce the augmented matrix $[A \;\; \mathbf{b}]$: $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

$$\begin{bmatrix} 1 & -3 & -4 & b_1 \\ -3 & 2 & 6 & b_2 \\ 5 & -1 & -8 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & b_2+3b_1 \\ 0 & 14 & 12 & b_3-5b_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & b_2+3b_1 \\ 0 & 0 & 0 & b_3-5b_1+2(b_2+3b_1) \end{bmatrix} = \begin{bmatrix} \textcircled{1} & -3 & -4 & b_1 \\ 0 & \textcircled{-7} & -6 & b_2+3b_1 \\ 0 & 0 & 0 & b_1+2b_2+b_3 \end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $b_1 + 2b_2 + b_3 = 0$. The set of such **b** is a plane through the origin in $\mathbf{R}^3$.

22. Row reduce the matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$ to determine whether it has a pivot in each row.

$$\begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -1 \\ -2 & 8 & -5 \end{bmatrix} \sim \begin{bmatrix} \textcircled{-2} & 8 & -5 \\ 0 & \textcircled{-3} & -1 \\ 0 & 0 & \textcircled{4} \end{bmatrix}$$

The matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$ has a pivot in each row, so the columns of the matrix span $\mathbb{R}^4$, by Theorem 4. That is, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^4$.

26. The equation in x_1 and x_2 involves the vectors **u**, **v**, and **w**, and it may be viewed as $[\mathbf{u} \quad \mathbf{v}]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{w}$. By definition of a matrix-vector product, $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{w}$. The stated fact that $3\mathbf{u} - 5\mathbf{v} - \mathbf{w} = \mathbf{0}$ can be rewritten as $3\mathbf{u} - 5\mathbf{v} = \mathbf{w}$. So, a solution is $x_1 = 3$, $x_2 = -5$.

1.5

6. $$\begin{bmatrix} 1 & 3 & -5 & 0 \\ 1 & 4 & -8 & 0 \\ -3 & -7 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 4 & 0 \\ 0 & \textcircled{1} & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \textcircled{x_1} \quad + 4x_3 &= 0 \\ \textcircled{x_2} - 3x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

The variable x_3 is free, $x_1 = -4x_3$, and $x_2 = 3x_3$.

In parametric vector form, the general solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4x_3 \\ 3x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$.

8. $$\begin{bmatrix} 1 & -2 & -9 & 5 & 0 \\ 0 & 1 & 2 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & -5 & -7 & 0 \\ 0 & \textcircled{1} & 2 & -6 & 0 \end{bmatrix}. \quad \begin{aligned} \textcircled{x_1} \quad - 5x_3 - 7x_4 &= 0 \\ \textcircled{x_2} + 2x_3 - 6x_4 &= 0 \end{aligned}$$

The basic variables are x_1 and x_2, with x_3 and x_4 free. Next, $x_1 = 5x_3 + 7x_4$ and $x_2 = -2x_3 + 6x_4$. The general solution in parametric vector form is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5x_3 + 7x_4 \\ -2x_3 + 6x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 7x_4 \\ 6x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 7 \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

12. $$\begin{bmatrix} 1 & 5 & 2 & -6 & 9 & 0 & 0 \\ 0 & 0 & 1 & -7 & 4 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 2 & -6 & 9 & 0 & 0 \\ 0 & 0 & 1 & -7 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 5 & 0 & 8 & 1 & 0 & 0 \\ 0 & 0 & \textcircled{1} & -7 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \textcircled{x_1} + 5x_2 \quad + 8x_4 + x_5 &= 0 \\ \textcircled{x_3} - 7x_4 + 4x_5 &= 0 \\ \textcircled{x_6} &= 0 \\ 0 &= 0 \end{aligned}$$

The basic variables are x_1, x_3, and x_6; the free variables are x_2, x_4, and x_5. The general solution is $x_1 = -5x_2 - 8x_4 - x_5$, $x_3 = 7x_4 - 4x_5$, and $x_6 = 0$. In parametric vector form, the solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -5x_2 - 8x_4 - x_5 \\ x_2 \\ 7x_4 - 4x_5 \\ x_4 \\ x_5 \\ 0 \end{bmatrix} = \begin{bmatrix} -5x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -8x_4 \\ 0 \\ 7x_4 \\ x_4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_5 \\ 0 \\ -4x_5 \\ 0 \\ x_5 \\ 0 \end{bmatrix} = x_2\begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -8 \\ 0 \\ 7 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

20. The line through **a** parallel to **b** can be written as $\mathbf{x} = \mathbf{a} + t\mathbf{b}$, where t represents a parameter:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} + t\begin{bmatrix} -7 \\ 8 \end{bmatrix}, \text{ or } \begin{cases} x_1 = 3 - 7t \\ x_2 = -4 + 8t \end{cases}$$

24. a. False. A nontrivial solution of $A\mathbf{x} = \mathbf{0}$ is any nonzero $\mathbf{x}$ that satisfies the equation. See the sentence before Example 2.

b. True. See Example 2 and the paragraph following it.

c. True. If the zero vector is a solution, then $\mathbf{b} = A\mathbf{x} = A\mathbf{0} = \mathbf{0}$.

d. True. See the paragraph following Example 3.

e. False. The statement is true only when the solution set of $A\mathbf{x} = \mathbf{0}$ is nonempty. Theorem 6 applies only to a consistent system.

26. (*Geometric argument using Theorem* 6.) Since $A\mathbf{x} = \mathbf{b}$ is consistent, its solution set is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$, by Theorem 6. So the solution set of $A\mathbf{x} = \mathbf{b}$ is a single vector if and only if the solution set of $A\mathbf{x} = \mathbf{0}$ is a single vector, and that happens if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(*Proof using free variables.*) If $A\mathbf{x} = \mathbf{b}$ has a solution, then the solution is unique if and only if there are no free variables in the corresponding system of equations, that is, if and only if every column of A is a pivot column. This happens if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.