6. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -9 \\ 11 \end{bmatrix}$, and the solution is $\mathbf{x} = A^{-1}\mathbf{b}$. To compute this by hand, the arithmetic is simplified by keeping the fraction $1/\det(A)$ in front of the matrix for A^{-1}. (The *Study Guide* comments on this in its discussion of Exercise 7.) From Exercise 3,

$$\mathbf{x} = A^{-1}\mathbf{b} = -\frac{1}{5}\begin{bmatrix} -5 & -5 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} -9 \\ 11 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -10 \\ 25 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}.$$ Thus $x_1 = 2$ and $x_2 = -5$.

10. **a.** False. The product matrix is invertible, but the product of inverses should be in the *reverse* order. See Theorem 6(b).

b. True, by Theorem 6(a).

c. True, by Theorem 4.

d. True, by Theorem 7.

e. False. The last part of Theorem 7 is misstated here.

16. Let $C = AB$. Then $CB^{-1} = ABB^{-1}$, so $CB^{-1} = AI = A$. This shows that A is the product of invertible matrices and hence is invertible, by Theorem 6.

Note: The *Study Guide* warns against using the formula $(AB)^{-1} = B^{-1}A^{-1}$ here, because this formula can be used only when both A and B are already known to be invertible.

20. **a.** Left-multiply both sides of $(A - AX)^{-1} = X^{-1}B$ by X to see that B is invertible because it is the product of invertible matrices.

b. Invert both sides of the original equation and use Theorem 6 about the inverse of a product (which applies because X^{-1} and B are invertible):

$$A - AX = (X^{-1}B)^{-1} = B^{-1}(X^{-1})^{-1} = B^{-1}X$$

Then $A = AX + B^{-1}X = (A + B^{-1})X$. The product $(A + B^{-1})X$ is invertible because A is invertible. Since X is known to be invertible, so is the other factor, $A + B^{-1}$, by Exercise 16 or by an argument similar to part (a). Finally,

$$(A + B^{-1})^{-1}A = (A + B^{-1})^{-1}(A + B^{-1})X = X$$

Note: This exercise is difficult. The algebra is not trivial, and at this point in the course, most students will not recognize the need to verify that a matrix is invertible.

22. Suppose A is invertible. By Theorem 5, the equation $A\mathbf{x} = \mathbf{b}$ has a solution (in fact, a unique solution) for each $\mathbf{b}$. By Theorem 4 in Section 1.4, the columns of A span $\mathbf{R}^n$.

28. When row 3 of A is replaced by $\text{row}_3(A) - 4\cdot\text{row}_1(A)$, write the result as

$$\begin{bmatrix} \text{row}_1(A) \\ \text{row}_2(A) \\ \text{row}_3(A) - 4\cdot\text{row}_1(A) \end{bmatrix} = \begin{bmatrix} \text{row}_1(I)\cdot A \\ \text{row}_2(I)\cdot A \\ \text{row}_3(I)\cdot A - 4\cdot\text{row}_1(I)\cdot A \end{bmatrix}$$

$$= \begin{bmatrix} \text{row}_1(I)\cdot A \\ \text{row}_2(I)\cdot A \\ [\text{row}_3(I) - 4\cdot\text{row}_1(I)]\cdot A \end{bmatrix} = \begin{bmatrix} \text{row}_1(I) \\ \text{row}_2(I) \\ \text{row}_3(I) - 4\cdot\text{row}_1(I) \end{bmatrix} A = EA$$

Here, E is obtained by replacing $\text{row}_3(I)$ by $\text{row}_3(I) - 4\cdot\text{row}_1(I)$.

4. The matrix $\begin{bmatrix} -7 & 0 & 4 \\ 3 & 0 & -1 \\ 2 & 0 & 9 \end{bmatrix}$ obviously has linearly dependent columns (because one column is zero), and so the matrix is not invertible (or singular) by (e) in the IMT.

12. **a.** True. If statement (k) of the IMT is true, then so is statement (j).

b. True. If statement (e) of the IMT is true, then so is statement (h).

c. True. See the remark immediately following the proof of the IMT.

d. False. The first part of the statement is not part (i) of the IMT. In fact, if A is any $n \times n$ matrix, the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps ^{n} into ^{n}, yet not every such matrix has n pivot positions.

e. True, by the IMT. If there is a **b** in ^{n} such that the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, then statement (g) of the IMT is false, and hence statement (f) is also false. That is, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ cannot be one-to-one.

Note: The solutions below for Exercises 13–30 refer mostly to the IMT. In many cases, however, part or all of an acceptable solution could also be based on various results that were used to establish the IMT.

14. If A is lower triangular with nonzero entries on the diagonal, then these n diagonal entries can be used as pivots to produce zeros below the diagonal. Thus A has n pivots and so is invertible, by the IMT. If one of the diagonal entries in A is zero, A will have fewer than n pivots and hence be singular.

Notes: For Exercise 14, another correct analysis of the case when A has nonzero diagonal entries is to apply the IMT (or Exercise 13) to A^T. Then use Theorem 6 in Section 2.2 to conclude that since A^T is invertible so is its transpose, A. You might mention this idea in class, but I recommend that you not spend much time discussing A^T and problems related to it, in order to keep from making this section too lengthy. (The transpose is treated infrequently in the text until Chapter 6.)

If you do plan to ask a test question that involves A^T and the IMT, then you should give the students some extra homework that develops skill using A^T. For instance, in Exercise 14 replace "columns" by "rows." Also, you could ask students to explain why an $n \times n$ matrix with linearly independent columns must also have linearly independent rows.

18. By (g) of the IMT, C is invertible. Hence, each equation $C\mathbf{x} = \mathbf{v}$ has a unique solution, by Theorem 5 in Section 2.2. This fact was pointed out in the paragraph following the proof of the IMT.

24. No conclusion about the columns of L may be drawn, because no information about L has been given. The equation $L\mathbf{x} = \mathbf{0}$ *always* has the trivial solution.

26. If the columns of A are linearly independent, then since A is square, A is invertible, by the IMT. So A^2, which is the product of invertible matrices, is invertible. By the IMT, the columns of A^2 span $\mathbf{R}^n$.

34. The standard matrix of T is $A = \begin{bmatrix} 6 & -8 \\ -5 & 7 \end{bmatrix}$, which is invertible because $\det A = 2 \neq 0$. By Theorem 9, T is invertible, and $T^{-1}(\mathbf{x}) = B\mathbf{x}$, where $B = A^{-1} = \frac{1}{2}\begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix}$. Thus

$$T^{-1}(x_1, x_2) = \frac{1}{2}\begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left(\frac{7}{2}x_1 + 4x_2, \frac{5}{2}x_1 + 3x_2\right)$$

4. Expanding along the first row:

$$\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 5\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 1(-2) - 3(1) + 5(5) = 20$$

Expanding along the second column:

$$\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix} = (-1)^{1+2} \cdot 3\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + (-1)^{2+2} \cdot 1\begin{vmatrix} 1 & 5 \\ 3 & 2 \end{vmatrix} + (-1)^{3+2} \cdot 4\begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix} = -3(1) + 1(-13) - 4(-9) = 20$$

6. Expanding along the first row:

$$\begin{vmatrix} 5 & -2 & 4 \\ 0 & 3 & -5 \\ 2 & -4 & 7 \end{vmatrix} = 5\begin{vmatrix} 3 & -5 \\ -4 & 7 \end{vmatrix} - (-2)\begin{vmatrix} 0 & -5 \\ 2 & 7 \end{vmatrix} + 4\begin{vmatrix} 0 & 3 \\ 2 & -4 \end{vmatrix} = 5(1) + 2(10) + 4(-6) = 1$$

10. First expand along the second row, then expand along either the third row or the second column of the remaining matrix.

$$\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix} = (-1)^{2+3} \cdot 3\begin{vmatrix} 1 & -2 & 2 \\ 2 & -6 & 5 \\ 5 & 0 & 4 \end{vmatrix}$$

$$= (-3)\left((-1)^{3+1} \cdot 5\begin{vmatrix} -2 & 2 \\ -6 & 5 \end{vmatrix} + (-1)^{3+3} \cdot 4\begin{vmatrix} 1 & -2 \\ 2 & -6 \end{vmatrix} \right) = (-3)(5(2) + 4(-2)) = -6$$

or

$$\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix} = (-1)^{2+3} \cdot 3\begin{vmatrix} 1 & -2 & 2 \\ 2 & -6 & 5 \\ 5 & 0 & 4 \end{vmatrix}$$

$$= (-3)\left((-1)^{1+2} \cdot (-2)\begin{vmatrix} 2 & 5 \\ 5 & 4 \end{vmatrix} + (-1)^{2+2} \cdot (-6)\begin{vmatrix} 1 & 2 \\ 5 & 4 \end{vmatrix} \right) = (-3)\big(2(-17) - 6(-6)\big) = -6$$

26. Since the matrix is triangular, by Theorem 2 the determinant is the product of the diagonal entries:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{vmatrix} = (1)(1)(1) = 1$$

28. Since the matrix is triangular, by Theorem 2 the determinant is the product of the diagonal entries:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{vmatrix} = (1)(k)(1) = k$$

30. A cofactor expansion along row 1 gives

$$\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

40. **a**. False. See Theorem 1.

b. False. See Theorem 2.