10. $$\begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ -2 & -6 & 2 & 3 & 9 \\ 3 & 7 & -3 & 8 & -7 \\ 3 & 5 & 5 & 2 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & -2 & 0 & 8 & -1 \\ 0 & -4 & 8 & 2 & 13 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & 0 & -4 & 7 & -7 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} =$$

$$-\begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & 0 & -4 & 7 & -7 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = -(-24) = 24$$

14. First use a row replacement to create zeros in the third column, and then expand down the third column:

$$\begin{vmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -3 & 4 & -2 & 8 \\ 3 & -4 & 0 & 4 \end{vmatrix} = \begin{vmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -9 & 0 & 0 & 0 \\ 3 & -4 & 0 & 4 \end{vmatrix} = 1\begin{vmatrix} 1 & 3 & -3 \\ -9 & 0 & 0 \\ 3 & -4 & 4 \end{vmatrix}$$

Now expand along the second row:

$$1\begin{vmatrix} 1 & 3 & -3 \\ -9 & 0 & 0 \\ 3 & -4 & 4 \end{vmatrix} = 1(-(-9))\begin{vmatrix} 3 & -3 \\ -4 & 4 \end{vmatrix} = (1)(9)(0) = 0$$

20. $$\begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$$

26. Since $\begin{vmatrix} 3 & 2 & -2 & 0 \\ 5 & -6 & -1 & 0 \\ -6 & 0 & 3 & 0 \\ 4 & 7 & 0 & -3 \end{vmatrix} = 0$, the columns of the matrix form a linearly dependent set.

28. **a.** True. See Theorem 3.

b. False. See the paragraphs following Example 2.

c. False. See Example 3.

d. False. See Theorem 5.

30. Suppose the two rows of a square matrix A are equal. By swapping these two rows, the matrix A is not changed so its determinant should not change. But since swapping rows changes the sign of the determinant, $\det A = -\det A$. This is only possible if $\det A = 0$. The same may be proven true for columns by applying the above result to A^T and using Theorem 5.

34. By Theorem 6 and Exercise 31,

$$\begin{aligned} \det(PAP^{-1}) &= (\det P)(\det A)(\det P^{-1}) = (\det P)(\det P^{-1})(\det A) \\ &= (\det P)\left(\frac{1}{\det P}\right)(\det A) = 1 \det A \\ &= \det A \end{aligned}$$

36. By Theorem 6 $\det A^4 = (\det A)^4$. Since $\det A^4 = 0$, then $(\det A)^4 = 0$. Thus $\det A = 0$, and A is not invertible by Theorem 4.

40. **a.** By Theorem 6, $\det AB = (\det A)(\det B) = -1 \times 2 = -2$.

b. By Theorem 6, $\det B^5 = (\det B)^5 = 2^5 = 32$.

c. By Exercise 32, $\det 2A = 2^4 \det A = 16 \times -1 = -16$.

d. By Theorems 5 and 6, $\det A^T A = (\det A^T)(\det A) = (\det A)(\det A) = -1 \times -1 = 1$.

e. By Theorem 6 and Exercise 31,
$\det B^{-1}AB = (\det B^{-1})(\det A)(\det B) = (1/\det B)(\det A)(\det B) = \det A = -1$.

2. **a.** If $\mathbf{u}=\begin{bmatrix} x \\ y \end{bmatrix}$ is in W, then the vector $c\mathbf{u}=c\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} cx \\ cy \end{bmatrix}$ is in W because $(cx)(cy)=c^2(xy)\geq 0$ since $xy\geq 0$.

b. *Example:* If $\mathbf{u}=\begin{bmatrix} -1 \\ -7 \end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then $\mathbf{u}$ and $\mathbf{v}$ are in W but $\mathbf{u}+\mathbf{v}$ is not in W.

5. Yes. Since the set is $\text{Span}\{t^2\}$, the set is a subspace by Theorem 1.

6. No. The zero vector is not in the set.

12. The set $W=\text{Span}\{\mathbf{u},\mathbf{v}\}$, where $\mathbf{u}=\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix}$. Thus W is a subspace of $\mathbb{R}^4$ by Theorem 1.

14. The augmented matrix is found as in Exercise 13c. Since

$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 2 & 4 \\ -1 & 3 & 6 & 7 \end{bmatrix}\sim\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

the equation $x_1\mathbf{v}_1+x_2\mathbf{v}_2+x_3\mathbf{v}_3=\mathbf{w}$ has no solution, and $\mathbf{w}$ is not in the subspace spanned by $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$.

16. Since the zero vector is not in W, W is not a vector space.

20. **a.** The following facts about continuous functions must be shown.

1. The constant function $\mathbf{f}(t)=0$ is continuous.
2. The sum of two continuous functions is continuous.
3. A constant multiple of a continuous function is continuous.

b. Let $H=\{\mathbf{f}\text{ in }C[a,b]\text{: }\mathbf{f}(a)=\mathbf{f}(b)\}$.

1. Let $\mathbf{g}(t)=0$ for all t in $[a,b]$. Then $\mathbf{g}(a)=\mathbf{g}(b)=0$, so $\mathbf{g}$ is in H.
2. Let $\mathbf{g}$ and $\mathbf{h}$ be in H. Then $\mathbf{g}(a)=\mathbf{g}(b)$ and $\mathbf{h}(a)=\mathbf{h}(b)$, and $(\mathbf{g}+\mathbf{h})(a)=\mathbf{g}(a)+\mathbf{h}(a)=\mathbf{g}(b)+\mathbf{h}(b)=(\mathbf{g}+\mathbf{h})(b)$, so $\mathbf{g}+\mathbf{h}$ is in H.
3. Let $\mathbf{g}$ be in H. Then $\mathbf{g}(a)=\mathbf{g}(b)$, and $(c\mathbf{g})(a)=c\mathbf{g}(a)=c\mathbf{g}(b)=(c\mathbf{g})(b)$, so $c\mathbf{g}$ is in H.

Thus H is a subspace of $C[a,b]$.

32. Both H and K contain the zero vector of V because they are subspaces of V. Thus the zero vector of V is in $H\cap K$. Let $\mathbf{u}$ and $\mathbf{v}$ be in $H\cap K$. Then $\mathbf{u}$ and $\mathbf{v}$ are in H. Since H is a subspace $\mathbf{u}+\mathbf{v}$ is in H. Likewise $\mathbf{u}$ and $\mathbf{v}$ are in K. Since K is a subspace $\mathbf{u}+\mathbf{v}$ is in K. Thus $\mathbf{u}+\mathbf{v}$ is in $H\cap K$. Let $\mathbf{u}$ be in $H\cap K$. Then $\mathbf{u}$ is in H. Since H is a subspace $c\mathbf{u}$ is in H. Likewise $\mathbf{v}$ is in K. Since K is a subspace $c\mathbf{u}$ is in K. Thus $c\mathbf{u}$ is in $H\cap K$ for any scalar c, and $H\cap K$ is a subspace of V.

The union of two subspaces is not in general a subspace. For an example in ^{2} let H be the x-axis and let K be the y-axis. Then both H and K are subspaces of ^{2}, but $H\cup K$ is not closed under vector addition. The subset $H\cup K$ is thus not a subspace of ^{2}.

33. **a.** Given subspaces H and K of a vector space V, the zero vector of V belongs to $H + K$, because $\mathbf{0}$ is in both H and K (since they are subspaces) and $\mathbf{0} = \mathbf{0} + \mathbf{0}$. Next, take two vectors in $H + K$, say $\mathbf{w}_1 = \mathbf{u}_1 + \mathbf{v}_1$ and $\mathbf{w}_2 = \mathbf{u}_2 + \mathbf{v}_2$ where $\mathbf{u}_1$ and $\mathbf{u}_2$ are in H, and $\mathbf{v}_1$ and $\mathbf{v}_2$ are in K. Then

$$\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{u}_1 + \mathbf{v}_1 + \mathbf{u}_2 + \mathbf{v}_2 = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{v}_1 + \mathbf{v}_2)$$

because vector addition in V is commutative and associative. Now $\mathbf{u}_1 + \mathbf{u}_2$ is in H and $\mathbf{v}_1 + \mathbf{v}_2$ is in K because H and K are subspaces. This shows that $\mathbf{w}_1 + \mathbf{w}_2$ is in $H + K$. Thus $H + K$ is closed under addition of vectors. Finally, for any scalar c,

$$c\mathbf{w}_1 = c(\mathbf{u}_1 + \mathbf{v}_1) = c\mathbf{u}_1 + c\mathbf{v}_1$$

The vector $c\mathbf{u}_1$ belongs to H and $c\mathbf{v}_1$ belongs to K, because H and K are subspaces. Thus, $c\mathbf{w}_1$ belongs to $H + K$, so $H + K$ is closed under multiplication by scalars. These arguments show that $H + K$ satisfies all three conditions necessary to be a subspace of V.

b. Certainly H is a subset of $H + K$ because every vector $\mathbf{u}$ in H may be written as $\mathbf{u} + \mathbf{0}$, where the zero vector $\mathbf{0}$ is in K (and also in H, of course). Since H contains the zero vector of $H + K$, and H is closed under vector addition and multiplication by scalars (because H is a subspace of V), H is a subspace of $H + K$. The same argument applies when H is replaced by K, so K is also a subspace of $H + K$.

6. First find the general solution of $A\mathbf{x}=\mathbf{0}$ in terms of the free variables. Since

$$[A\quad \mathbf{0}] \sim \begin{bmatrix} 1 & 0 & 6 & -8 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

the general solution is $x_1 = -6x_3 + 8x_4 - x_5$, $x_2 = 2x_3 - x_4$, with x_3, x_4, and x_5 free. So

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

and a spanning set for Nul A is

$$\left\{ \begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

10. The set W is the set of all solutions to the homogeneous system of equations $a + 3b - c = 0$, $a + b + c - d = 0$. Thus W = Nul A, where $A = \begin{bmatrix} 1 & 3 & -1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix}$. Thus W is a subspace of ^{4} by Theorem 2, and is a vector space.

12. The set W is a subset of ^{4}. If W were a vector space (under the standard operations in ^{4}), then it would be a subspace of ^{4}. But W is not a subspace of ^{4} since the zero vector is not in W. Thus W is not a vector space.

16. An element in this set may be written as

$$b\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 1 \\ 5 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} b \\ c \\ d \end{bmatrix}$$

where b, c and d are any real numbers. So the set is Col A where $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix}$.

28. Let A be the coefficient matrix of the given systems of equations. Since the first system has a solution, the constant vector $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 9 \end{bmatrix}$ is in Col$A$. Since Col A is a subspace of ^{3}, it is closed under scalar multiplication. Thus $5\mathbf{b} = \begin{bmatrix} 0 \\ 5 \\ 45 \end{bmatrix}$ is also in Col A, and the second system of equations must thus have a solution.

30. Since $T(\mathbf{0}_V) = \mathbf{0}_W$, the zero vector $\mathbf{0}_W$ of W is in the range of T. Let $T(\mathbf{x})$ and $T(\mathbf{w})$ be typical elements in the range of T. Then since $T(\mathbf{x}) + T(\mathbf{w}) = T(\mathbf{x}+\mathbf{w}), T(\mathbf{x}) + T(\mathbf{w})$ is in the range of T and the range of T is closed under vector addition. Let c be any scalar. Then since $cT(\mathbf{x}) = T(c\mathbf{x}), cT(\mathbf{x})$ is in the range of T multiplication. Hence the range of T is a subspace of W.

32. Any quadratic polynomial $\mathbf{q}$ for which $\mathbf{q}(0)=0$ will be in the kernel of T. The polynomial $\mathbf{q}$ must then be $\mathbf{q}=at+bt^2$. Thus the polynomials $\mathbf{p}_1(t)=t$ and $\mathbf{p}_2(t)=t^2$ span the kernel of T. If a vector is in the range of T, it must be of the form $\begin{bmatrix} a \\ a \end{bmatrix}$. If a vector is of this form, it is the image of the polynomial $\mathbf{p}(t)=a$ in $_2$. Thus the range of T is $\left\{\begin{bmatrix} a \\ a \end{bmatrix} : a \text{ real}\right\}$.