1. Suppose $A, B \in \mathbb{C}^{n \times n}$ are simultaneously diagonalizable (i.e., there exists a nonsingular matrix $S$ such that $S^{-1}AS$ and $S^{-1}BS$ are both diagonal). Show that $A, B$ commute (i.e., $AB = BA$).

2. Let $A, S \in \mathbb{C}^{n \times n}$ be such that $S$ is nonsingular and $S^{-1}AS$ is diagonal. If $S = QR$ is a QR-factorization (i.e., $Q$ is unitary, $R$ is upper triangular), show that the matrix $T := Q^H AQ$ is upper triangular and hence $A = QTQ^H$ gives a Schur decomposition.

3. Let $A \in \mathbb{C}^{n \times n}$ be a matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Show that $A$ is normal if and only if

$$\text{trace}(AA^H) = \sum_{i=1}^{n} |\lambda_i|^2.$$ 

4. Suppose $A, B \in \mathbb{C}^{n \times n}$ commute (i.e., $AB = BA$) and $A$ has $n$ distinct eigenvalues. Show that $B$ is diagonalizable.

5. Let $A \in \mathbb{R}^{n \times n}$ be a matrix that has a complex eigenvalue $\lambda = a + \sqrt{-1}b$ ($a, b \in \mathbb{R}, b \neq 0$), with a complex eigenvector $u + \sqrt{-1}v$ ($u, v \in \mathbb{R}^n$). Show that $u, v$ are linearly independent.