Math 202A HW3

1. As $A, B$ are simultaneously diagonalizable, there exists $S$ such that $A' = S^{-1}AS, B' = S^{-1}BS$ are both diagonal. Thus

\[ AB = (SA'S^{-1})(SB'S^{-1}) = SA'B'S^{-1} = SB'A'S^{-1} = BA \] (1)

The equality $A'B' = B'A'$ holds because $A', B'$ are both diagonal.

2. Suppose $A' = S^{-1}AS$ is a diagonal matrix. Then

\[ A' = S^{-1}AS = R^{-1}Q^HDQ \Rightarrow Q^H = RA'R^{-1} \] (3)

$R$ is upper triangular $\Rightarrow R^{-1}$ is also upper triangular. Thus $T = RA'R^{-1}$ must be upper triangular because the product of upper triangular matrices is still upper triangular.

3. $\Rightarrow$: $A$ is normal $\Leftrightarrow$ there exists unitary matrix $Q$ and diagonal matrix $D = \text{diag}\{\lambda_1, \ldots, \lambda_n\}$ such that $A = Q^HDQ$. Thus $AA^H = Q^HDD^HQ \Rightarrow \text{trace}(AA^H) = \text{trace}(DD^H)$. $DD^H = \text{diag}\{|\lambda_1|^2, \ldots, |\lambda_n|^2\} \Rightarrow$ trace$(AA^H) = \text{trace}(DD^H) = \sum_{i=1}^n|\lambda_i|^2$.

$\Leftarrow$: Let $C = Q^HAQ$ be a Schur decompostion of $A$ where $Q$ is a unitary matrix and $C$ is upper triangular. Thus we have,

\[ AA^H = QCC^HQ^H \] (4)
\[ A^HA = QC^HCQ^H \] (5)

Therefore $A$ is normal iff $C$ is normal. We may write $C$ as,

\[
\begin{pmatrix}
\lambda_1 & c_{12} & \cdots & c_{1n} \\
0 & \lambda_2 & \cdots & c_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{pmatrix}
\]

Then we have $(CC^H)_{kk} = \sum_{i=k}^{n}|c_{ki}|^2$ and $(C^HC)_{kk} = \sum_{i=1}^{k}|c_{ki}|^2$ where $c_{kk} = \lambda_k$ for $k = 1, \ldots, n$. Additionally, $CC^H = Q^HAA^HQ$
is similar to $AA^H$ which means that trace$(CC^H) = trace(AA^H) = \sum_{i=1}^{n} |\lambda_i|^2$ Then we have,
\[
trace(CC^H) = \sum_{k=1}^{n} (CC^H)_{kk} = \sum_{k=1}^{n} \sum_{i=k}^{n} |c_{ki}|^2 \\
= \sum_{k=1}^{n} |\lambda_k|^2 + \sum_{k=1}^{n-1} \sum_{i=k+1}^{n} |c_{ki}|^2 = \sum_{k=1}^{n} |\lambda_k|^2 \\
\Rightarrow c_{ki} = 0, \ i \neq k
\]
Thus $C$ is a diagonal matrix and therefore normal. By the conclusion that $C$ is normal $\Rightarrow A$ is normal, we have $A$ is normal.

4. $A$ has $n$ distinct eigenvalues $\Rightarrow$ there exists nonsigular matrix $S$ such that $A' = S^{-1}AS$ is a diagonal matrix with diagonal entries being $\{\lambda_1, \ldots, \lambda_n\}$ which is the set of eigenvalues of $A$.
\[
SA'S^{-1}B = AB = BA = BSA'S^{-1} \\
\Leftrightarrow A'(S^{-1}BS) = (S^{-1}BS)A'
\]
Thus it is equal to prove $B' = S^{-1}BS$ is diagonalizable. Let $B' = (b_{ij})_{1 \leq i,j \leq n}$. $(A'B')_{ij} = \lambda_i b_{ij}$ and $(B'A')_{ij} = \lambda_j b_{ij}$. $A'B' = B'A' \Rightarrow (\lambda_i - \lambda_j)b_{ij} = 0, \forall 1 \leq i,j \leq n$. And $\lambda_i \neq \lambda_j$ when $i \neq j \Rightarrow b_{ij} = 0$, if $i \neq j$. Thus $B'$ is diagonal and $B$ is diagonalizable.

5. If $u, v$ are linear dependent, we may assume $u = \alpha v, \alpha \in \mathbb{R}$ and $v \neq 0$. Then $u + \sqrt{-1}v = (\alpha + \sqrt{-1})v$.
\[
(\alpha + \sqrt{-1})Av = A(u + \sqrt{-1}v) = \lambda(u + \sqrt{-1}v) = \lambda(\alpha + \sqrt{-1})v \\
\Rightarrow Av = \lambda v = av + \sqrt{-1}bv
\]
But $\text{Im}(Av) = 0 \neq bv = \text{Im}(\lambda v) = \text{Im}(Av)$ which is a contradiction. Therefore $u, v$ are linear independent.