1. If \( \lambda \) is an eigenvalue of \( A \), then \( \lambda^2 \) is an eigenvalue of \( A^2 \). By \( A = A^2 \), we know that \( \lambda = \lambda^2 \Rightarrow \lambda = 0 \) or 1. Thus there are two large Jordan blocks \( J'_0, J'_1 \) corresponding to eigenvalue 0, 1 respectively. Let \( J_i \) be any small block of \( J'_i, i = 0, 1 \). \( A^2 = A \) and \( (A - I)^2 = A^2 - 2A + I = I - A \Rightarrow J'_0^2 = J_0 \) and \( (J_1 - I_1)^2 = -J_1 + I_1 \). We know that

\[
J_i - \lambda_i I_i = (a_{ij})_{1 \leq i,j \leq n_i} = \begin{cases} 1 & j = i + 1 \\
0 & \text{otherwise} \end{cases}
\]

\[
(J_i - \lambda_i I_i)^2 = (b_{ij})_{1 \leq i,j \leq n_i} = \begin{cases} 1 & j = i + 2 \\
0 & \text{otherwise} \end{cases}
\]

Thus \( J'_0^2 = J_0, (J_1 - I_1)^2 = I_1 - J_1 \Rightarrow J_0, J_1 \) are both 1 \( \times \) 1 blocks. \( \text{rank}(A) = r \Rightarrow J'_0 \) is a \((n - r) \times (n - r)\) block and \( J'_1 \) is a \( r \times r \) block.

Due to the above proof, we know that each small block of \( J'_0, J'_1 \) has dimension 1 \( \times \) 1, so we get

\[
J = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}
\]  \hspace{1cm} (1)

2. \( \Rightarrow \): \( A \) is similar to \( B \), there exists a nonsingular matrix \( Q \) such that \( Q^{-1}BQ = A \). Assume \( J \) is the Jordan normal form of \( A \), then there exists \( R \) such that \( R^{-1}AR = J \). Thus \( R^{-1}Q^{-1}BQR = (QR)^{-1}BQR = J \) which means that \( J \) is also the Jordan normal form of \( B \).

\( \Leftarrow \): \( J \) is the Jordan normal form of \( A \) and \( B \), then there exists nonsingular \( Q, R \) such that

\[
Q^{-1}AQ = R^{-1}BR = J \Rightarrow (QR)^{-1}AQR^{-1} = B
\]  \hspace{1cm} (2)

Therefore \( A \) is similar to \( B \).

3. If \( \text{rank}(A) = n \), then \( \text{rank}(A^k) = n, \forall k \geq 1 \). Thus the claim holds.

Now assume \( \text{rank}(A) < n \). As \( \text{rank}(A^{k+1}) \leq \text{rank}(A^k) \), there must be a \( k \leq n \) such that \( A^k = A^{k+1} \). Otherwise,

\[
\text{rank}(A) > \text{rank}(A^2) > \ldots > \text{rank}(A^n) > \text{rank}(A^{n+1})
\]

\( \Rightarrow \text{rank}(A^n+1) \leq \text{rank}(A) - n < 0 \)
which is certainly impossible. Therefore there exists $k \leq n$ such that $\text{rank}(A^k) = \text{rank}(A^{k+1})$. So we have

$$\text{range}(A^k) = \text{range}(A^{k+1}) \Leftrightarrow \text{range}(A^{k+1}) = \{Ax | x \in \text{range}(A^k)\} = \text{range}(A^k)$$

$$\Rightarrow \text{range}(A^{k+2}) = \{Ax | x \in \text{range}(A^{k+1}) = \text{range}(A^k)\} = \text{range}(A^{k+1})$$

$$\Rightarrow \text{range}(A^{k+2}) = \text{range}(A^{k+1})$$

By induction,

$$\text{rank}(A^k) = \text{rank}(A^{k+1}) = \text{rank}(A^{k+2}) = \cdots \quad (3)$$

4. Suppose $v, Av, \ldots, A^{k-1}v$ are linearly dependent, then there exists $\lambda_0, \ldots, \lambda_{k-1}$ such that $\Pi_{i=0}^{k-1} \lambda_i \neq 0$ and $\sum_{i=0}^{k-1} \lambda_i A^i v = 0$. Let $m$ be the smallest number such that $\lambda_m \neq 0$. Because there must be at least one $\lambda_i \neq 0$, such an $m$ exists. Additionally $A^k v = 0 \Rightarrow A^i v = 0, \forall i \geq k$.

We then have

$$\sum_{i=0}^{k-1} \lambda_i A^i v = \sum_{i=m}^{k-1} \lambda_i A^i v = 0$$

$$\Rightarrow A^{k-m-1} \sum_{i=m}^{k-1} \lambda_i A^i v = \sum_{i=m}^{k-1} \lambda_i A^{i+k-m-1} = \lambda_m A^{k-1} v = 0$$

$$\Rightarrow A^{k-1} v = 0 \quad \text{or} \quad \lambda_m = 0.$$ 

But according to the provided assumption and condition, we know that $A^{k-1} v \neq 0$ and $\lambda_m \neq 0$. Thus $v, Av, \ldots, A^{k-1}v$ are linearly independent.

5. $\text{Det}(A - \lambda I) = (1 - \lambda I)^4 = 0 \Rightarrow \lambda = 1$. Thus $A$ has only one eigenvalue $1$. $\text{rank}(A - I) = 3 \Rightarrow \text{dim null}(A - I) = 1$. Therefore the geometric multiplicity of $\lambda$ is 1. $J$ has only one Jordan block and has form

$$J = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4)$$

Assume nonsingular matrix $P = (p_1, p_2, p_3, p_4)$ satisfies that $AP = PJ$ where $p_i \in \mathbb{C}^n$. It is equal to

$$(Ap_1, Ap_2, Ap_3, Ap_4) = (p_1, p_1 + p_2, p_2 + p_3, p_3 + p_4) \quad (5)$$

$$\Rightarrow (A - I)^i p_i = 0 \quad \text{and} \quad (A - I)^{i-1} p_i \neq 0$$
By solving linear equations, we can get \( \text{null}(A - I)^i = \text{span}\{e_1, \ldots, e_i\} \Rightarrow e_i \in \text{span}\{e_i\} \). We can choose \( p_4 = e_4 \), then according to equation (5), we can get \( p_3 = 4e_3, p_2 = 12e_2, p_1 = 24e_1 \).

\[
P = \begin{pmatrix}
24 & 0 & 0 & 0 \\
0 & 12 & 0 & 0 \\
0 & 0 & 4 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]  \hspace{1cm} (6)

\( J = P^{-1}AP \).