1. For all positive integers \( p \geq n \geq 1 \), show that the function
\[
f(x) := (x_1 \cdot x_2 \cdots x_n)^{1/p} \quad \text{where} \quad x = (x_1, \ldots, x_n)
\]
is concave over the positive orthant \( \mathbb{R}^n_{++} \).

2. For a matrix \( X \in \mathbb{R}^{n \times n} \), denote by \( \sigma_i(X) \) the \( i \)th largest singular value of \( X \). For each \( k = 1, \ldots, n \), show that the sum function \( f(X) = \sigma_1(X) + \cdots + \sigma_k(X) \) is convex.

3. Find the expression for the conjugate function of
\[
f(x) := \max\{x_1, \ldots, x_n\} \quad \text{where} \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.
\]

4. Consider the quadratic optimization
\[
\begin{align*}
\min \quad & 2x_1x_2 + 3x_2x_3 + 5x_1x_3 \\
\text{s.t.} \quad & x_2^2 + 2x_2^3 + 3x_3^2 \leq 1.
\end{align*}
\]
Formulate the dual optimization problem. Compute the optimizers for both the primal and dual. Does the strong duality hold?

5. Consider the quadratic optimization problem
\[
\begin{align*}
\min \quad & x_1^2 + x_2^2 \\
\text{s.t.} \quad & x_2^2 + x_1x_2 \geq 1, x_1x_2 + x_2^2 \geq 1.
\end{align*}
\]
Formulate the dual optimization problem. Compute the optimizers for both the primal and dual. Does the strong duality hold?