Lecture 2  Review of Eigenvalues

1  General Eigenvalues

Let $A \in \mathbb{C}^{n \times n}$. A number $\lambda \in \mathbb{C}$ is an eigenvalue of $A$ if there exists $0 \neq x \in \mathbb{C}^n$ such that

$$Ax = \lambda x.$$ 

The $x$ is called an eigenvector of $A$ associated with $\lambda$. The eigenvalues of $A$ are the roots to the characteristic polynomial of $A$:

$$p(\lambda) = \det(A - \lambda I_n).$$

where $I$ is the $n \times n$ identity matrix. The matrix $A$ is singular (resp. nonsingular) if and only if it has (resp. has no) zero eigenvalues.

If $p(\lambda)$ has the factorization $c \cdot (\lambda_1 - \lambda)^{m_1} \cdots (\lambda_k - \lambda)^{m_k}$ for distinct $\lambda_i$, we say $\lambda_i$ is an eigenvalue of $A$ with multiplicity $m_i$, which is also called the algebraic multiplicity of $\lambda_i$. The dimension of the eigenspace

$$\mathcal{N}(A - \lambda_i I) = \{ u : (A - \lambda_i I)u = 0 \}$$

is called the geometric multiplicity of $\lambda_i$, and is denoted $\rho_i$. We always have $\rho_i \leq m_i$.

**Theorem 1** (Jordan’s Canonical Form). For any $A \in \mathbb{C}^{n \times n}$, there exists an invertible $P \in \mathbb{C}^{n \times n}$ such that $P^{-1}AP = \text{diag}(J_1, \ldots, J_r)$ where each $J_i$ is a Jordan block

$$J_i = \begin{bmatrix}
\lambda_i & 1 & 0 & \cdots & 0 \\
\lambda_i & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\lambda_i & 1 & & & \\
\lambda_i & & & & \\
\end{bmatrix}$$

If every $J_i$ has dimension $1 \times 1$, then $A$ is called diagonalizable, which is true if and only if every eigenvalue has same algebraic and geometric multiplicities.

Let $A, B \in \mathbb{C}^{n \times n}$ be two square matrices. They are similar to each other, denoted by $A \sim B$, if there exists invertible $P$ such that

$$A = PBP^{-1}.$$ 

- If $A \sim B$, then they the same set of eigenvalues.
- The $A$ and $B$ are similar if and only if they have the same Jordan’s Canonical Form.
Theorem 2 (Shur’s Canonical Form). For any $A \in \mathbb{C}^{n \times n}$, there exists a unitary matrix $U \in \mathbb{C}^{n \times n}$ ($U^*U = UU^* = I_n$) such that

$$U^*AU = \begin{bmatrix} R_{11} & R_{12} & R_{13} & \cdots & R_{1n} \\ R_{21} & R_{22} & R_{23} & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ R_{n-1,1} & R_{n-1,2} & \cdots & R_{n-1,n-1} & R_{n-1,n} \\ R_{n,1} & R_{n,2} & \cdots & \cdots & R_{n,n} \end{bmatrix}$$

is upper triangular. Here each $R_{ii}$ is an eigenvalue of $A$.

2 Symmetric eigenvalues

A real matrix $A \in \mathbb{R}^{n \times n}$ is symmetric if $A = A^T$.

Theorem 3. Suppose $A \in \mathbb{R}^{n \times n}$ is symmetric. Then we have

- All the eigenvalues are real.
- Eigenvectors for distinct eigenvalues are orthogonal to each other.
- There exists an orthogonal matrix $Q$ such that

$$Q^T AQ = \text{diag}(\lambda_1, \ldots, \lambda_n)$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$.

For a symmetric $A \in \mathbb{R}^{n \times n}$, denote by $\lambda_i(A)$ the $i$-th biggest eigenvalue of $A$.

Theorem 4. Suppose $A \in \mathbb{R}^{n \times n}$ is symmetric. Then we have

- $\lambda_1(A) = \max_{x \neq 0} \frac{x^T Ax}{x^T x}$, and $\lambda_n(A) = \min_{x \neq 0} \frac{x^T Ax}{x^T x}$.
- Max-Min and Min-Max characterizations for $\lambda_i(A)$:

$$\lambda_i(A) = \min_{\dim(S) = n-i+1} \max_{0 \neq x \in S} \frac{x^T Ax}{x^T x}.$$

$$\lambda_i(A) = \max_{\dim(S) = i} \min_{0 \neq x \in S} \frac{x^T Ax}{x^T x}.$$

- Each eigenvalue of $A$ is a critical value of optimization problem

$$\min \quad x^T Ax \quad \text{subject to} \quad x^T x = 1.$$
If a symmetric $A \in \mathbb{R}^{n \times n}$ has eigenvalues such that
\[
\lambda_1(A) \geq \cdots \geq \lambda_r(A) > 0 = \lambda_{r+1}(A) = \cdots = \lambda_{r+s}(A) > \lambda_{r+s+1}(A) \geq \cdots \geq \lambda_{r+s+t}(A),
\]
we say the triple $(r, s, t)$ is the inertia of $A$, and $r-t$ is the signature of $A$.

For two symmetric $A, B$, if there exists invertible $P$ such that $A = P^T BP$, we say $A$ is congruent to $B$.

**Theorem 5.** If $A, B$ are symmetric and congruent to each other, then they have the same inertia and signature.

**Proof.** Use the min-max or max-min characterization of eigenvalues. \hfill \Box

### 3 Singular values

Let $A \in \mathbb{R}^{m \times n}$ be a generally nonsquare matrix, then there exist orthogonal matrices $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ such that
\[
U^T AV = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_r)
\]
where $\sigma_1 \geq \cdots \geq \sigma_r > 0$ are called singular values of $A$. If we write
\[
U = \begin{bmatrix} u_1 & \cdots & u_m \end{bmatrix}, \quad V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix},
\]
we say $u_i$ is a left singular vector of $A$, $v_j$ is a right singular vector of $A$. The relationship between eigenvalues and singular values is
\[
U^T AA^T U = \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix}, \quad U^T A^T A U = \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix},
\]
\[
\begin{bmatrix} 0 & V \\ U & 0 \end{bmatrix}^T \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} 0 & V \\ U & 0 \end{bmatrix} = \begin{bmatrix} 0 & \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} & 0 \end{bmatrix}.
\]
The Moore-Penrose pseudoinverse of $A$ is defined as
\[
A^+ = V \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T.
\]
They satisfy the relations
\[
(AA^+)^T = AA^+, \quad (A^+ A)^T = A^+ A, \quad AA^+ A = A, \quad A^+ AA^+ = A^+, \quad (A^+)^+ = A.
\]
The Moore-Penrose pseudoinverse of $A$ is unique.