Lecture 3 Positive Semidefinite Matrices

1 Definitions and Characterizations

Definition 1. *A symmetric matrix $A \in S\mathbb{R}^{n\times n}$ is called positive semidefinite if $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n$, and is called positive definite if $x^T Ax > 0$ for all nonzero $x \in R^n$. The set of positive semidefinite matrices is denoted $\mathcal{S}^n_+$, and the set of positive definite matrices is denoted by $\mathcal{S}^n_{++}$. The cone $\mathcal{S}^n_+$ is a proper cone (i.e., closed, convex, pointed, and solid).*

If A is positive semidefinite (resp. positive definite), we denote $A \succeq 0$ (resp. $A \succ 0$).

Theorem 2. *The following statements are equivalent:*

- *The symmetric matrix A is positive semidefinite.*
- *All eigenvalues of A are nonnegative.*
- *All the principal minors of A are nonnegative.*
- *There exists B such that $A = B^T B$.*

Theorem 3. *The following statements are equivalent:*

- *The symmetric matrix A is positive definite.*
- *All eigenvalues of A are positive.*
- *All the leading principal minors of A are positive.*
- *There exists nonsingular square matrix B such that $A = B^T B$.*

Theorem 4. *Let $A \in S\mathbb{R}^{n\times n}$. Then A is positive semidefinite if and only if all the coefficients of its characteristic polynomial*

$$p(\lambda) \quad = \quad \det(\lambda I_n - A) = \lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_1\lambda + p_0$$

has alternating signs, i.e., $(-1)^{n-i}p_i \geq 0$ for all i.

This theorem is implied by the following lemma.

Lemma 5. *Suppose the monic univariate polynomial $p(t) = t^n + p_{n-1}t^{n-1} + \cdots + p_1 t + p_0$ has only real roots. Then, all its roots are nonpositive if and only if all coefficients are nonnegative.*

Proof. If all the roots t_i of $p(t)$ are nonpositive, then from the factorization

$$p(t) = \prod_{i=1}^{n}(t - t_i)$$

we immediately know $p(t)$ has nonnegative coefficients. □

2 Properties

If $X - Y \succeq 0$, then we write $X \succeq Y$.

Theorem 6. *For two symmetric X, Y, if $X \succeq Y$, then*

$$\lambda_i(X) \geq \lambda_i(Y), \text{ for every } i.$$

Here $\lambda_i(\cdot)$ denotes the i-th largest eigenvalue.

Proof. Use the characterization of $\lambda_i(\cdot)$. □

Congruent transformations preserve positive semidefiniteness.

Theorem 7. *Let P be a nonsingular matrix.*

- *The $A \succeq 0$ if and only if $P^T AP \succeq 0$.*
- *The $A \succ 0$ if and only if $P^T AP \succ 0$.*

If P is singular, then $A \succeq 0$ implies $P^T AP \succeq 0$ (while the reverse might not).

Theorem 8 (Schur's complement)**.** *Let $A \succ 0$. Then*

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0 \quad \Longleftrightarrow \quad C - B^T A^{-1} B \succeq 0.$$

Theorem 9. *If matrices A, B satisfy*

$$\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \succeq 0,$$

then $A \succeq 0$ and $B = 0$.

A matrix $A \in \mathbb{R}^{n \times n}$ is **stable** if the real parts of all its eigenvalues are negative.

Theorem 10. *A matrix A is stable if and only if there exists a symmetric positive definite P such that*

$$PA + A^T P \prec 0.$$

Proof. Apply Schur's theorem. □

Definition 11. *Let $A, B \in \mathbb{R}^{n \times n}$. Their Hadamard product is*

$$A \circ B = (A_{ij} B_{ij}).$$

Theorem 12 (The Schur Product Theorem)**.** *If $A, B \in S\mathbb{R}^{n \times n}$ are positive semidefinite, then their Hadamard product $A \circ B$ is also positive semidefinite. Moreover, if both A and B are positive definite, then $A \circ B$ is also positive definite.*

Proof. Since $A, B \succeq 0$, we can write

$$A = u_1 u_1^T + \cdots + u_n u_n^T, \quad B = v_1 v_1^T + \cdots + v_n v_n^T.$$

Observe that

$$A \circ B = \sum_{i,j=1}^{n} w_{ij} w_{ij}^T, \quad \text{where} \quad w_{ij} = u_i \circ v_j.$$

So we know $A \circ B \succeq 0$.

If both $A, B \succ 0$, then $A \circ B$ must be positive definite. Otherwise suppose there exists $x \in \mathbb{R}^n$ such that

$$x^T A \circ B x = \sum_{i,j=1}^{n} (w_{ij}^T x)^2 = \sum_{i,j=1}^{n} (x^T u_i \circ v_j)^2 \sum_{i,j=1}^{n} (u_i (x \circ v_j))^2 = 0.$$

This implies that

$$u_i (x \circ v_j) = 0, \qquad \forall\, i, j.$$

Since $u_1, \ldots, u_n$ are LID, we must have

$$x \circ v_j = 0, \qquad j = 1, \ldots, n.$$

The above then implies

$$\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} x = 0.$$

Since $v_1, \ldots, v_n$ are LID, we have $x = 0$. Therefore, $A \circ B$ must be positive definite. $\square$