Lecture 5  Semidefinite Programming I

A standard semidefinite programming (SDP) problem is

\[ (P): \min_X \quad C \bullet X \]
\[ \text{s.t.} \quad A_i \bullet X = b_i, \quad i = 1, \ldots, m \]
\[ X = X^T \succeq 0 \]

where all \( A_i \in \mathbb{S}_R^{n \times n}, b \in \mathbb{R}^m, C \in \mathbb{S}_R^{n \times n} \), \( \bullet \) denotes the standard Frobenius inner product, and \( X \succeq 0 \) means \( X \) is positive semidefinite.

- When all \( A_i \) and \( C \) are diagonal, \( (P) \) reduces to LP.
- When \( X \) is not symmetric, it is equivalent to symmetric case.
- Problem \( (P) \) is convex, but its boundary is typically highly nonlinear.

The dual problem of \( (P) \) is

\[ (D): \max_y \quad b^T y \]
\[ \text{s.t.} \quad C - \sum_{i=1}^m A_i \succeq 0. \]

**Example 1.** Consider the primal problem

\[ \min_X \quad I \bullet X \]
\[ \text{s.t.} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bullet X = 2, \quad X = X^T \succeq 0 \]

Its dual is

\[ \max_y \quad 2y \]
\[ \text{s.t.} \quad \begin{bmatrix} 1 & -y \\ -y & 1 \end{bmatrix} \succeq 0. \]

**Example 2.** Consider the primal problem

\[ \min_X \quad \sum_{i \neq j} X_{ij} \]
\[ \text{s.t.} \quad X_{11} = \cdots X_{nn} = 1, \quad X = X^T \succeq 0 \]

Its dual is

\[ \max_y \quad y_1 + \cdots + y_n \]
\[ \text{s.t.} \quad \begin{bmatrix} -y_1 & 1 & \cdots & 1 \\ 1 & -y_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & -y_n \end{bmatrix} \succeq 0. \]

The condition that \( e^T X e \geq 0 \) implies that any feasible \( X \) satisfy

\[ \sum_{i \neq j} X_{ij} \geq -n. \]

So the optimal value is \(-n\).
Theorem 3 (Weak Duality). For any $X$ feasible for (P) and $y$ feasible for (D), it holds that

$$C \cdot X \geq b^T y.$$ 

- (P) can be formulated in (D), and (D) can be formulated in (P).
- If (P) is unbounded from below, (D) is not feasible.
- If (D) is unbounded from above, (P) is not feasible.
- Both (P) and (D) would be infeasible simultaneously.

Here are some examples of SDP.

1. **Minimizing the max eigenvalue** Let $A(x)$ be an affine symmetric matrix

$$A(x) = A_0 + x_1 A_1 + \cdots x_m A_m,$$

where every $A_i \in \mathbb{S}^{n \times n}$ is symmetric. We want to find a vector $x$ such that $\lambda_{\text{max}}(A(x))$ is minimum. This problem is equivalent to the dual SDP

$$\max_x -x_{m+1}$$

$$\text{s.t. } A_0 + x_1 A_1 + \cdots x_m A_m - x_{m+1} I_n \succeq 0.$$

2. **Minimizing the 2-norm of matrices** Let $A(x)$ be an affine non-symmetric matrix

$$A(x) = A_0 + x_1 A_1 + \cdots x_m A_m,$$

where every $A_i \in \mathbb{R}^{n \times k}$. We want to find a vector $x$ such that $\|A(x)\|_2$ is minimum. This problem is equivalent to the dual SDP

$$\max_x -x_{m+1}$$

$$\text{s.t. } \begin{bmatrix} x_{m+1} I_n & A(x) \\ A(x)^T & x_{m+1} I_k \end{bmatrix} \succeq 0, \quad \text{diag}(c - Ax) \succeq 0.$$ 

3. **A quasi-convex fractional programming** Consider problem

$$\min_{x \in \mathbb{R}^n} \frac{(b^T x + \beta)^2}{a^T x + \alpha}$$

$$\text{s.t. } Ax \leq c$$

where we assume $a^T x + \alpha \geq 0$ for all $x$ satisfying $Ax \leq c$. This problem is equivalent to the dual SDP

$$\max_{x,t} -t$$

$$\text{s.t. } \begin{bmatrix} t & b^T x + \beta \\ b^T x + \beta & a^T x + \alpha \end{bmatrix} \succeq 0, \quad \text{diag}(c - Ax) \succeq 0.$$
4. Convex quadratically constrained programming  Consider problem

$$\begin{align*}
\min_x & \quad d^T x \\
\text{s.t.} & \quad x^T A_i^T A_i x + b_i^T x + c_i \leq 0, \quad i = 1, \ldots, m.
\end{align*}$$

This problem is equivalent to the dual SDP

$$\begin{align*}
\max_{x,t} & \quad -t \\
\text{s.t.} & \quad \begin{bmatrix} t - c_i - b_i^T x & A_i x \\ x^T A_i^T & I \end{bmatrix} \succeq 0, \quad i = 1, \ldots, m.
\end{align*}$$

5. Control Theory  We want to find $P \succ 0$ whose condition number while satisfying

$$A_i^T P + P A_i \preceq 0, \quad i = 1, \ldots, m.$$ 

Here all $A_i$ are stable matrices (the real parts of eigenvalues of $A_i$ are negative). This problem is equivalent to the dual SDP

$$\begin{align*}
\max_{x,t} & \quad -t \\
\text{s.t.} & \quad A_i^T P + P A_i \preceq 0, \quad i = 1, \ldots, m. \\
& \quad t I \succeq P \succeq I.
\end{align*}$$