

Lecture 12  Nonnegative Univariate Polynomials

1 Definitions

Definition 1. A univariate polynomial \( p(x) \) is positive semidefinite (psd) or nonnegative if the evaluation \( p(x) \geq 0 \) for all real \( x \).

Definition 2. A univariate polynomial \( p(x) \) is a sum of squares (SOS) if there exist real polynomials \( q_1(x), \ldots, q_r(x) \) such that
\[
p(x) = q_1(x)^2 + \cdots + q_r(x)^2.
\]

Theorem 3. If a univariate polynomial \( p(x) \) is nonnegative, then \( p(x) \) must be SOS (actually it must be a sum of two squares).

Proof. Apply the factorization of \( p(x) \).

2 Properties

Let \( P_{2d} \) be the set defined as
\[
P_{2d} = \{ f = (f_0, \ldots, f_{2d}) : f_0 + f_1x + \cdots + f_{2d}x^{2d} \geq 0 \forall x \in \mathbb{R} \}.
\]

Theorem 4. The set \( P_{2d} \) is a proper cone (closed, convex, pointed, and solid).

For a given vector \( y = (y_0, y_1, \ldots, y_{2d}) \in \mathbb{R}^{2d+1} \), it defines a Hankel matrix as
\[
H_d(y) = \begin{bmatrix}
y_0 & y_1 & \cdots & y_{d-1} \\
y_1 & y_2 & \cdots & y_d \\
\vdots & \vdots & \ddots & \vdots \\
y_{d-1} & y_d & \cdots & y_{2d}
\end{bmatrix}.
\]

Theorem 5. The dual cone of \( P_{2d} \) is \( M_{2d} = \{ y \in \mathbb{R}^{2d+1} : H_d(y) \succeq 0 \} \).

Proof. Suppose \( y \in P_{2d}^* \), then
\[
f^T y \geq 0 \quad \forall y \in P_{2d}.
\]

Clearly, \( M_{2d} \) is a closed convex set. If \( y \notin M_{2d} \), then there exists a separating hyperplane \( a^T z = b \) such that
\[
a^T z \geq b, \forall z \in M_{2d}, \quad a^T y < b.
\]

Clearly, \( b \leq 0 \), and we can choose \( b = 0 \) because \( P_{2d} \) is homogeneous. In particular, we choose \( z = (1, x, \ldots, x^{2d}) \), which belongs to \( M_{2d} \) for any \( x \), then
\[
a(x) = a_0 + a_1x + \cdots + a_{2d}x^{2d} \geq 0, \quad \forall x \in \mathbb{R}.
\]

So \( a \in P_{2d} \). But \( a^T y < 0 \) contradicts this fact. Thus \( y \in M_{2d} \).

The proof of the other direction is obvious.
3 Nonnegative polynomials over an interval

Let $I \subset (-\infty, +\infty)$ be a closed interval. Define a set

$$ P_m(I) = \{ f(x) \in \mathbb{R}[x]_m : f(x) \geq 0 \forall x \in I \}. $$

**Theorem 6.** The cone $P_m(I)$ can be characterized as follows.

- If $I = [a, +\infty)$, then $f(x) \in P_{2d+1}(I)$ if and only if
  $$ f(x) = p(x) + (x - a) \cdot q(x) $$
  for some $p(x), q(x) \in \mathbb{R}[x]_{2d}$ being SOS.
- If $I = [a, b]$ and $m = 2d$, then $f(x) \in P_{2d}(I)$ if and only if
  $$ f(x) = p(x) + q(x)(b - x)(x - a) $$
  for some $p(x) \in \mathbb{R}[x]_{2d}$ and $q(x) \in \mathbb{R}[x]_{2d-2}$ being SOS.
- If $I = [a, b]$ and $m = 2d + 1$, then $f(x) \in P_{2d+1}(I)$ if and only if
  $$ f(x) = r(x) + (x - a)p(x) + q(x)(b - x) $$
  for some $p(x), q(x), r(x) \in \mathbb{R}[x]_{2d}$ being SOS.

**Proof.** Let $c_1, \ldots, c_n$ be the roots of $f(x)$, and factor $f(x)$ as

$$ f(x) = f_n(x - c_1) \cdots (x - c_r) \cdot (x - c_{r+s})(x - c_{r+s}^*)(x - c_{r+1})(x - c_{r+1}^*). $$

Here $c_1 \leq \cdots \leq c_r \in \mathbb{R}$ and $c_{r+1}, \ldots, c_n \in \mathbb{C}/\mathbb{R}$. Note that each $(x - c_{r+i})(x - c_{r+i}^*)$ is a sum of two squares. For $c_i (1 \leq i \leq r)$, if $c_i \in \text{int}(I)$, then $c_i$ must have even multiplicity. Thus we can write $f(x)$ generally as

$$ f(x) = (g(x)^2 + h(x)^2) \cdot (-1)^{k_1}(x - c_1) \cdots (-1)^{k_r}(x - c_r) $$

with each $(-1)^{k_i}(x - c_i) \geq 0$ on $I$. Without loss of generality, just consider $a = 0, b = 1$.

- When $I = [0, +\infty)$, every $c_i \leq 0$. For $r = 2$, we get
  $$ f(x) = (g(x)^2 + h(x)^2) \cdot ((x - c_1)^2 - c_1(c_2 - c_1) + x(c_2 - c_1)). $$
  For $r > 2$, the representation is similar.

- When $I = [0, 1]$ and $m$ is even, then every $c_i \leq 0$ or $c_i \geq 1$. Note the identities
  $$ 1 - x = (1 - x)^2 + (x - x^2), \quad x = x^2 + (x - x^2). $$
When \( c_i \leq c_j \leq 0 \), we get 
\[
(x - c_i)(x - c_j) = (x - c_j + c_j - c_i)(x - c_i) \\
= (x - c_j)^2 + (c_j - c_i)(x - c_i) \\
= (x - c_j)^2 - (c_j - c_i)c_i + (c_j - c_i)x \\
= (x - c_j)^2 - (c_j - c_i)c_i + (c_j - c_i)(x^2 + x(1 - x)).
\]

When \( c_j \geq c_i \geq 1 \), we get 
\[
(c_i - x)(c_j - x) = (c_i - x)(c_i - x + c_j - c_i) \\
= (c_i - x)^2 + (c_i - x)(c_j - c_i) \\
= (c_i - x)^2 + (c_i - 1)(c_j - c_i) + (1 - x)(c_j - c_i) \\
= (c_i - x)^2 + (c_i - 1)(c_j - c_i) + ((1 - x)^2 + x(1 - x))(c_j - c_i).
\]

When \( c_i \leq 0 \) and \( c_j \geq 1 \), we get 
\[
(x - c_i)(c_j - x) = (x - c_i)(c_j - 1 + 1 - x) \\
= x(1 - x) + x(c_j - 1) - c_i(1 - x) - c_i(c_j - 1) \\
= x(1 - x) + (x^2 + x(1 - x))(c_j - 1) - c_i((1 - x)^2 + x(1 - x) - c_i(c_j - 1)).
\]

The product of the above cases are always in the required form.

- When \( I = [0, 1] \) and \( m \) is odd, every \( c_i \leq 0 \) or \( c_i \geq 1 \).
  - If \( c_i \leq 0 \), we get 
    \[
    x - c_i = (-c_i) + x
    \]
  - If \( c_i \geq 1 \), we get 
    \[
    c_i - x = (c_i - 1) + (1 - x).
    \]
When we expand \((-1)^{k_1}(x - c_1) \cdots (-1)^{k_r}(x - c_r)\) by using the above, we get something like 
\[
h_0(x) + h_1(x)x + h_2(x)(1 - x) + h_3(x)x(1 - x).
\]
where \( h_i(x) \) are all SOS, the degree of each summand is at most \( r \). Using the identity 
\[
x(1 - x) = (1 - x)^2x + x^2(1 - x),
\]
we get required representation.