Lecture 15  SOS and PSD polynomials

1 Definitions

Let \( p(x) = \mathbb{R}[x] := \mathbb{R}[x_1, \ldots, x_n] \). We say \( p(x) \) is positive semidefinite (psd) or nonnegative if \( p(u) \geq 0 \) for every \( u \in \mathbb{R}^n \). We say \( p(x) \) is sum of squares (SOS) if there exist real polynomials \( q_1(x), \ldots, q_r(x) \) such that

\[
p(x) = q_1(x)^2 + \cdots + q_r(x)^2.
\]

Clearly, if \( p(x) \) is SOS, then \( p(x) \) is psd.

Example 1. (i) The polynomial \( 2x_1^4 + 2x_1^3x_2 - x_1^2x_2^2 + 5x_2^4 \) is SOS, equaling

\[
\frac{1}{2} \left( (2x_1^2 - 3x_2^2 + x_1x_2)^2 + (x_2^2 + 3x_1x_2)^2 \right).
\]

(ii) The polynomial \( x_1^4 - (2x_2x_3 + 1)x_1^2 + (x_2^2x_3^2 + 2x_2x_3 + 2) \) is SOS, equaling

\[
1 + x_1^2 + (1 - x_1^2 + x_2x_3)^2
\]

(iii) The polynomial \( 3(x_1^4 + x_2^4 + x_3^4 + x_4^4 - 4x_1x_2x_3x_4) \) is SOS, being

\[
(x_1^2 + x_3^2 - x_2 - x_4^2)^2 + (x_1^2 + x_2^2 - x_3 - x_4^2)^2 + (x_1^2 + x_2^2 - x_2 - x_3^2)^2 +
2(x_1x_2 - x_3x_4)^2 + 2(x_1x_3 - x_2x_4)^2 + 2(x_1x_4 - x_2x_3)^2.
\]

Denote two sets of polynomials

\[
P_{n,2d} = \{ f \in \mathbb{R}[x] : f(x) \text{ is psd of degree } 2d \},
\]

\[
\Sigma_{n,2d} = \{ f \in \mathbb{R}[x] : f(x) \text{ is SOS of degree } 2d \}.
\]

Theorem 2 (Hilbert, 1888). The equality \( P_{n,2d} = \Sigma_{n,2d} \) holds if and only if

- \( n = 1 \), or
- \( d = 1 \), or
- \( (n, 2d) = (2, 4) \).

Now we show a nonnegative polynomial that is not SOS. For example, consider the Motzkin polynomial

\[
M(x, y, z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2.
\]

Obviously \( M(x, y, z) \) is psd, but \( M(x, y, z) \) is not SOS. Otherwise suppose

\[
M(x, y, z) = \sum_k (A_kx^3 + B_kx^2y + C_kx^2z + D_kxy^2 + E_kxyz + F_kxz^2 + G_ky^3 + H_ky^2z + I_kyz^2 + J_kz^3)^2.
\]
Since \( M(x, y, z) \) does not have \( x^6, y^6 \), we know \( A_k = G_k = 0 \). So
\[
M(x, y, z) = \sum_k (B_k x^2 y + C_k x^2 z + D_k xy^2 + E_k xyz + F_k xz^2 + H_k y^2 z + I_k yz^2 + J_k z^3)^2.
\]
Since \( M(x, y, z) \) does not have \( x^4 z^2, y^4 z^2 \), we know \( C_k = H_k = 0 \). So
\[
M(x, y, z) = \sum_k (B_k x^2 y + D_k xy^2 + E_k xyz + F_k xz^2 + I_k yz^2 + J_k z^3)^2.
\]
Since \( M(x, y, z) \) does not have \( x^2 z^4, y^2 z^4 \), we know \( F_k = I_k = 0 \). So
\[
M(x, y, z) = \sum_k (B_k x^2 y + D_k xy^2 + E_k xyz + J_k z^3)^2.
\]
Hence, the coefficient of \( x^2 y^2 z^2 \) in \( M(x, y, z) \) is
\[
M(x, y, z) = \sum_k E_k^2 \geq 0.
\]
This is a contradiction !!!

## 2 How to test SOS membership

Let \( f(x) \in \mathbb{R}[x_1, \ldots, x_n]_{2d} \). Let \( [x]_d \) denote the column vector of all monomials of degree at most \( d \). Define symmetric matrices \( A_\alpha \) such that
\[
[x]_d^T [x]_d = \sum_{|\alpha| \leq 2d} A_\alpha x^\alpha.
\]
Then \( f(x) \) is SOS if and only if there exists \( X \succeq 0 \) such that
\[
f(x) \equiv [x]^T X [x]_d,
\]
or equivalently by comparing coefficients
\[
f_\alpha = A_\alpha \cdot X, \quad \forall |\alpha| \leq 2d.
\]
Testing whether \( f(x) \) is SOS would be done by
\[
\text{finding } X \succeq 0, \quad A_\alpha \cdot X = f_\alpha, \quad \forall |\alpha| \leq 2d.
\]
This is an SDP feasibility problem.

**Example 3.** The polynomial \( p(x) = 2x_1^4 + 2x_1^3 x_2 - x_1^2 x_2^2 + 5x_2^4 \) has representation
\[
\begin{bmatrix}
x_1^2 \\
x_2^2 \\
x_1 x_2
\end{bmatrix}^T
\begin{bmatrix}
2 & -\alpha & 1 \\
-\alpha & 5 & 0 \\
1 & 0 & -1 + 2\alpha
\end{bmatrix}
\begin{bmatrix}
x_1^2 \\
x_2^2 \\
x_1 x_2
\end{bmatrix}
\]
\[
p(x) \text{ is SOS } \iff \exists G \succeq 0
\]
When \( \alpha = 3 \), the Gram matrix \( G \) is positive semidefinite.
**Theorem 4.** The dual cone $\Sigma^*_{n,2d}$ is $\{y : M_d(y) \succeq 0\}$.

**Proof.** Suppose $y \in \Sigma^*_{n,2d}$, then

$$f^T y \geq 0, \quad \forall f \in \Sigma_{n,2d}. $$

Write $f(x) = [x]_d^T X [x]_d = X \bullet ([x]_d [x]^T_d)$, then comparing coefficients gives

$$f^T y = X \bullet ([x]_d [x]^T_d) = X \bullet M_d(y) \geq 0, \quad \forall X \succeq 0. $$

We clearly have $M_d(y) \succeq 0$.

The other direction would be obtained by applying Hahn-Banach Theorem. \hfill \Box

For instance, $n = 2, d = 2$, $\Sigma^*_{2,2d}$ consists of all $y$ such that $M_2(y) \succeq 0$, where

$$M_2(y) = \begin{bmatrix} y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{02} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{bmatrix}. $$

**Theorem 5.** The dual cone $P^*_{n,2d}$ is the closure of the following set

$$\left\{ y : M_d(y) \succeq 0, \text{ y has a measure} \right\}. $$

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