Lecture 16  Minimizing Multivariate Polynomials

1 SOS and SDP

How to test if a polynomial is SOS? Suppose \( f(x) \) has degree 2\( d \) and is SOS. Then there exist polynomials \( q_1(x), \ldots, q_r(x) \) of degree \( d \) such that
\[
f(x) = q_1(x)^2 + \cdots + q_r(x)^2.
\]
Write each \( q_i(x) = q_i^T[x]_d \). Then
\[
f(x) = (q_1^T[x]_d)^2 + \cdots + (q_r^T[x]_d)^2 = [x]_d^T X [x]_d
\]
where \( X = q_1 q_1^T + \cdots + q_r q_r^T \succeq 0 \).

**Theorem 1.** Let \( f(x) \) be a polynomial of degree 2\( d \). Then \( f(x) \) is SOS if and only if there exists \( X \succeq 0 \) such that
\[
f(x) = [x]_d^T X [x]_d.
\]

Let \( A_{\alpha} \) be constant symmetric matrices such that
\[
[x]_d[x]_d^T = \sum_{|\alpha| \leq 2d} A_{\alpha} x^\alpha.
\]
Write \( f(x) \) as
\[
f(x) = \sum_{|\alpha| \leq 2d} f_{\alpha} x^\alpha.
\]
So \( f(x) \) is SOS if and only if
\[
\exists X \succeq 0 : A_{\alpha} \cdot X = f_{\alpha}, \quad \forall \alpha.
\]
This is precisely an SDP feasibility problem.

2 Minimizing Polynomials

Let \( f(x) \in \mathbb{R}[x] \) of degree 2\( d \). Consider problem
\[
f_{\text{min}} = \min_{x \in \mathbb{R}^n} f(x).
\]
It is NP-hard to find \( f_{\text{min}} \). It is equivalent to
\[
f_{\text{min}} = \max_{\text{s.t.}} \gamma \quad f(x) - \gamma \geq 0 \quad \forall x \in \mathbb{R}^n.
\]
Replacing psd by SOS, we get the relaxation
\[
f_{\text{sos}} = \max_{\text{s.t.}} \gamma \quad f(x) - \gamma \text{ is SOS}.
\]
The constraint above is $f(x) - \gamma \in \Sigma_{n,2d}$. If we write $f(x) = \sum_{|\alpha| \leq 2d} f_\alpha x^\alpha$, the above is equivalent to the SDP

$$(P): \begin{cases} f_{\text{sos}} = \max \gamma \\ \text{s.t. } A_0 \cdot X + \gamma = f_0, \\ A_\alpha \cdot X = f_\alpha, \quad \forall \alpha \neq 0. \end{cases}$$

The dual of the above SDP is

$$(D): \begin{cases} f_{\text{mom}} = \min \sum_\alpha f_\alpha y_\alpha \\ \text{s.t. } M_d(y) \succeq 0, y_0 = 1. \end{cases}$$

**Theorem 2.** (1) The dual problem (D) has nonempty interior.
(2) (P) and (D) have the same optimal value, that is, $f_{\text{sos}} = f_{\text{mom}} \leq f_{\text{min}}$.
(3) (P) has a minimizer, that is, $f_{\text{sos}}$ is attainable in (P), while (D) might not achieve its max value.
(4) If $y^*$ is a maximizer of (D) and rank $M_d(y^*) = 1$, then $f_{\text{sos}} = f_{\text{mom}} = f_{\text{min}}$, and a global minimizer of $f(x)$ can be obtained.

**Example 3.**

$$f(x, y) = 4x^2 - \frac{21}{10} x^4 + \frac{1}{3} x^6 + xy - 4y^2 + 4y^4$$

$f(x, y)$ is nonconvex and has several local minimizers.

SOS relax. gives exact lower bound

$$f^* = f_{\text{sos}}^* \approx -1.03$$

which is attained at

$(x_1^*, y_1^*) \approx (0.09, -0.71)$ $(x_2^*, y_2^*) \approx (-0.09, 0.71)$