Lecture 22  Positive Polynomials

1 Positive Polynomials

Let $K \subset \mathbb{R}^n$ be a basic closed semialgebraic set, i.e., there exists polynomials $g_1(x), \ldots, g_m(x)$ such that

$$K = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}.$$  

A polynomial $f(x)$ is called nonnegative on $K$ if

$$f(u) \geq 0 \quad \forall u \in K.$$  

A polynomial $f(x)$ is called positive on $K$ if

$$f(u) > 0 \quad \forall u \in K.$$  

We are interested in sufficient and/or necessary conditions for a polynomial $f(x)$ positive or nonnegative on a set $K$.

Definition 1. The preordering of polynomials $g_1(x), \ldots, g_m(x)$ is the set

$$P(g_1, \ldots, g_m) = \left\{ \sum_{\nu \in \{0,1\}^m} \sigma_\nu(x)g_1(x)^{\nu_1} \cdots g_m(x)^{\nu_m} : \text{each } \sigma_\nu \text{ is SOS} \right\}$$

The quadratic module of polynomials $g_1(x), \ldots, g_m(x)$ is the set

$$M(g_1, \ldots, g_m) = \left\{ \sum_{i=0}^m \sigma_i(x)g_i(x) : \text{each } \sigma_i \text{ is SOS} \right\}$$

Here $g_0(x) = 1$.

Theorem 2. The sets $P(g_1, \ldots, g_m)$ and $M(g_1, \ldots, g_m)$ are convex.

- If $f(x) \in P(g_1, \ldots, g_m)$, then $f(x)$ is nonnegative on $K$.
- If $f(x) \in M(g_1, \ldots, g_m)$, then $f(x)$ is nonnegative on $K$.

2 Schmüdgen’s and Putinar’s Positivstellensatz

Theorem 3 (Schmüdgen, 1991). Let $K = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}$ be a compact set. If $f(x)$ is positive on $K$, then $f(x) \in P(g_1, \ldots, g_m)$.

Example 4. The quadratic $f = x_1x_2 + 1$ is positive on $x^T x \leq 1$. We have

$$x_1x_2 + 1 = \frac{1}{2}(x_1 + x_2)^2 + \frac{1}{2} + \frac{1}{2}(1 - x_1^2 - x_2^2).$$
Example 5. If \( f(x) \) is only nonnegative on \( S \), then Schm"udgen’s theorem may not be true. For example, consider
\[
f(x) = 1 - x^2, \quad S = \{ (1 - x^3)^3 \geq 0 \}.
\]
Suppose there are SOS polynomials \( \sigma, \phi \) such that
\[
1 - x^2 = \sigma(x) + \phi(x)(1 - x^3)^3.
\]
Then \(-1\) must be a root of \( \sigma(x) \) with multiplicity 2, but it has multiplicity 1 on the left.

Theorem 6 (Putinar, 1993). Let \( K = \{ x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_m(x) \geq 0 \} \) be a compact set. Suppose there exists \( N > 0 \) such that
\[
N - \| x \|_2^2 \in M(g_1, \ldots, g_m).
\]
If \( f(x) \) is positive on \( K \), then \( f(x) \in M(g_1, \ldots, g_m) \).

Remark 7. The condition that \( N - \| x \|_2^2 \in P(g_1, \ldots, g_m) \) is called the archimedean condition (AC). When AC holds, the set \( K \) must be compact, but the reverse might not be true. When AC fails, the conclusion of Putinar’s theorem might not be true.

Example 8. Not every compact set is AC. For instance, consider
\[
S = \{ x \in \mathbb{R}^2 : x_1 - \frac{1}{2} \geq 0, x_2 - \frac{1}{2} \geq 0, 1 - x_1 x_2 \geq 0 \}.
\]
Clearly, it is compact. There are no SOS polynomials \( \sigma \) such that
\[
N - (x_1^2 + x_2^2) = \sigma_0 + (x_1 - \frac{1}{2})\sigma_1 + (x_2 - \frac{1}{2})\sigma_2 + (1 - x_1 x_2)\sigma_3.
\]
If they exist, then \( D = \max(\deg(\sigma_0, \sigma_3 + 2)) \geq 1 + \max(\deg(\sigma_1, \sigma_2)) \). When \( D = 2 \), it does not work. When \( D > 2 \), the highest even term of \( \sigma_0 + (1 - x_1 x_2)\sigma_3 \) must vanish, which is not possible.

- Schm"udgen’s Positivstellensatz has a weaker assumption than Putinar’s Positivstellensatz, but the conclusion is also weaker.
- When \( f(x) \) is only nonnegative but not strictly positive on \( K \), both Schm"udgen’s and Putinar’s Positivstellensatz may fail.
- When \( f(x) \) is only nonnegative on \( K \), then for any \( \epsilon > 0 \) we have \( f(x) + \epsilon \in M(g_1, \ldots, g_m) \) or \( P(g_1, \ldots, g_m) \). As \( \epsilon \to 0 \), typically the degree bounds of the representation of \( f(x) + \epsilon \) in \( M(g_1, \ldots, g_m) \) or \( P(g_1, \ldots, g_m) \) goes to infinity.

Example 9. For every \( \epsilon > 0 \), the polynomial \( 1 - x^2 + \epsilon \) is positive on
\[
S = \{ x \in \mathbb{R} : (1 - x^2)^3 \geq 0 \}.
\]
By Schm"udgen’s or Putinar’s Positivstellensatz, there exists SOS polynomials \( P_\epsilon, Q_\epsilon \) such that
\[
1 - x^2 + \epsilon = P_\epsilon(1 - x^2)^3 + Q_\epsilon.
\]
It was shown by Stengle that
\[
\deg(P_\epsilon), \deg(Q_\epsilon) \geq \mathcal{O}(\epsilon^{-1/2}).
\]