1. A matrix $A \in \mathbb{R}^{n \times n}$ is said to be stable if the real parts of all its eigenvalues are negative. Prove that a matrix $A$ is stable if and only if there exists a symmetric positive definite matrix $P$ such that 

$$PA + A^TP < 0.$$ 

2. Prove that the max-matrix norm defined as 

$$\|X\|_{\text{max}} = \max_{ij} |X_{ij}|$$ 

is not an operator norm.

3. Let $\| \cdot \|_F$ be the Frobenius norm. Prove that for any $X, Y \in \mathbb{R}^{n \times n}$ it holds 

$$\|XY\|_F \leq \|X\|_2\|Y\|_F.$$ 

4. Let $A \in \mathbb{R}^{m \times n}$. Prove that the Moore-Penrose pseudoinverse $A^+$ is a minimizer of the optimization 

$$\min_{X \in \mathbb{R}^{n \times m}} \|AX - I\|_F.$$ 

5. Let $X \in \mathbb{R}^{n \times n}$ be a symmetric positive semidefinite matrix such that $-1 \leq X_{ij} \leq 1$. Define $\arcsin(X)$ component-wisely as 

$$\arcsin(X) = (\arcsin(X_{ij}))_{1 \leq i,j \leq n}.$$ 

Prove that $\arcsin(X) \succeq X$. 
