1. Let $P$ be the following convex cone

$$P = \left\{ (f_0, f_1, f_2, f_3, f_4) : f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 \geq 0 \ \forall x \in [-1,1] \right\}.$$  

Find an SDP description for the dual cone $P^*$. 

2. Let $M(x, y, z)$ be the Motzkin polynomial

$$M(x, y, z) = x^4 y^2 + x^2 y^4 + z^6 - 3x^2 y^2 z^2.$$  

Show that $(x^2 + y^2 + z^2)M(x, y, z)$ is an SOS polynomial. (Hint: use SOSTOOLS).

3. Let $f(x)$ be the following bivariate polynomial

$$f(x) = x_1^4 + x_2^2 + x_3^6 - 3x_1^2 x_2^2.$$  

What is the global minimum $f_{\min}$ of this $f(x)$? What is the lower bound $f_{\text{sos}}$ returned by the standard SOS relaxation? Does $f_{\text{sos}} = f_{\min}$? (Hint: use Gloptipoly or SOSTOOLS).

4. Let $f(x)$ be a polynomial of degree 4 in $x \in \mathbb{R}^3$ bounded from below, with infinimum $f_{\min}$.

Let $f^*$ be the optimal value of the following SOS program

$$(P) : \left\{ \begin{array}{l} \max_{s.t.} \gamma \\
\ \ \ \ (1 + x^T x) \cdot (f(x) - \gamma) \in \Sigma_{3,6} \end{array} \right.$$  

Show that $f^* \leq f_{\min}$. Formulate the dual problem (D). Does the strong duality hold between (P) and (D)? Justify your answer.

5. Consider the dynamic system described as

$$\begin{cases} \dot{x}_1 = -x_1 - 2x_2^2, \\
\dot{x}_2 = -x_2 - x_1 x_2 - 2x_2^3. \end{cases}$$  

In the above, $x = x(t)$ is the state function. A quadratic form $V(x) = x^T A x$ with $A > 0$ is a Lyapunov function if

$$\frac{d}{dt} V(x(t)) \leq 0$$  

for all states $x = x(t)$ satisfying the above dynamic system. Formulate an SDP for how to find a Lyapunov function. Find an explicit Lyapunov function for this system.