Question 2.3.2:

\[ f(x) = -x^3 - \cos x \]
\[ f'(x) = -3x^2 + \sin x \]

Start: \( p_0 = -1 \):

\[ p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} = -0.89033 \]

\[ p_2 = p_1 - \frac{f(p_1)}{f'(p_1)} = -0.454 \]

Start: \( p_0 = 0 \):

Since \( f'(p_0) = 0 \): Newton fails.

Question 2.3.4(a):

2.3.4(a). Secant method:

\[ p_2 = p_1 - \frac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)} = -0.68507 \]

\[ p_3 = p_2 - \frac{f(p_2)(p_2 - p_0)}{f(p_2) - f(p_0)} = -1.25216 \]

Question 2.3.18:

Use Newton’s method to solve for roots of

\[ 0 = \frac{1}{2} + \frac{1}{4}x^2 - x \sin x - \frac{1}{2} \cos 2x \]

SOLUTION: Newton’s method with \( p_0 = \frac{\pi}{2} \) gives \( p_{15} = 1.895488 \) and with \( p_0 = 5\pi \) gives \( p_{19} = 1.895489 \). With \( p_0 = 10\pi \), the sequence does not converge in 200 iterations.

The results do not indicate the fast convergence usually associated with Newton’s method because the function and its derivative have the same roots. As we approach a root, we are dividing by numbers with small magnitude, which increases the round-off error.
Question 2.3.24

Determine the minimal annual interest rate \( i \) at which an amount \( P = \$1500 \) per month can be invested to accumulate an amount \( A = \$750,000 \) at the end of 20 years based on the annuity due equation

\[
A = \frac{P}{i} [(1 + i)^n - 1].
\]

SOLUTION: This is simply a root-finding problem where the function is given by

\[
f(i) = A - \frac{P}{i} \left[ (1 + i)^n - 1 \right] = 750000 - \frac{1500}{(i/12)} \left[ (1 + i/12)^{(12)(20)} - 1 \right].
\]

Notice that \( n \) and \( i \) have been adjusted because the payments are made monthly rather than yearly. The approximate solution to this equation can be found by any method in this section. Newton’s method is a bit cumbersome for this problem, since the derivative of \( f \) is complicated. The Secant method would be a likely choice. The minimal annual interest is approximately 6.67%.

Question 2.4.2: Apply Newton’s method

Question 2.4.6(a):

6. a. Show that the sequence \( p_n = 1/n \) converges linearly to \( p = 0 \), and determine the number of terms required to have \( |p_n - p| < 5 \times 10^{-2} \).

SOLUTION: First note that \( \lim_{n \to \infty} \frac{1}{n} = 0 \). Since

\[
\lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n \to \infty} \frac{1/(n+1)}{1/n} = \lim_{n \to \infty} \frac{n}{n+1} = 1,
\]

the convergence is linear. To have \( |p_n - p| < 5 \times 10^{-2} \), we need \( 1/n < 0.05 \), which implies that \( n > 20 \).

Question 2.4.8:

a. The sequence \( p_n = 10^{-2n} \) converges quadratically to zero;
b. The sequence \( p_n = 10^{-n^k} \) does not converge to zero quadratically, regardless of the size of \( k > 1 \).

SOLUTION;

a. Since

\[
\lim_{n \to \infty} \frac{|p_{n+1} - 0|}{|p_n - 0|^2} = \lim_{n \to \infty} \frac{10^{-2(n+1)}}{(10^{-2n})^2} = \lim_{n \to \infty} \frac{10^{-2n+1}}{10^{-2n}} = \lim_{n \to \infty} \frac{10^{-2n+1}}{10^{-2n+1}} = 1,
\]

the sequence is quadratically convergent.

b. For any \( k > 1 \),

\[
\lim_{n \to \infty} \frac{|p_{n+1} - 0|}{|p_n - 0|^2} = \lim_{n \to \infty} \frac{10^{-(n+1)^k}}{(10^{-n^k})^2} = \lim_{n \to \infty} \frac{10^{-(n+1)^k}}{10^{-2n^k}} = \lim_{n \to \infty} \frac{10^{2n^k-(n+1)^k}}{10^{-2n^k}}
\]

diverges. So the sequence \( p_n = 10^{-n^k} \) does not converge quadratically for any \( k > 1 \).
Question 2.4.10:

10. Show that the fixed-point method

\[ g(x) = x - \frac{mf(x)}{f'(x)} \]

has \( g'(p) = 0 \), if \( p \) is a zero of \( f \) of multiplicity \( m \).

**SOLUTION:** If \( f \) has a zero of multiplicity \( m \) at \( p \), then a function \( q(x) \) exists with

\[ f(x) = (x - p)^m q(x), \quad \text{where} \quad \lim_{x \to p} q(x) \neq 0. \]

Since

\[ f'(x) = m(x - p)^{m-1} q(x) + (x - p)^m q'(x), \]

we have

\[ g(x) = x - \frac{mf(x)}{f'(x)} = x - \frac{m(x - p)^m q(x)}{m(x - p)^{m-1} q(x) + (x - p)^m q'(x)}, \]

which reduces to

\[ g(x) = x - \frac{m(x - p)q(x)}{mq(x) + (x - p)q'(x)}. \]

Differentiating this expression and evaluating at \( x = p \) gives

\[ g'(p) = 1 - \frac{mq(p)[mq(p)]}{[mq(p)]^2} = 0. \]

If \( f''' \) is continuous, Theorem 2.9 implies that this sequence produces quadratic convergence once we are close enough to the solution \( p \).

Question 2.4.12:

12. Suppose that \( f \) has \( m \) continuous derivatives. Show that \( f \) has a zero of multiplicity \( m \) at \( p \) if and only if

\[ 0 = f(p) = f'(p) = \cdots = f^{(m-1)}(p), \quad \text{but} \quad f^{(m)}(p) \neq 0. \]
SOLUTION: If \( f \) has a zero of multiplicity \( m \) at \( p \), then \( f \) can be written as
\[
 f(x) = (x - p)^m q(x), \quad \text{for } x \neq p, \text{ where } \lim_{x \to p} q(x) \neq 0.
\]
Thus
\[
 f'(x) = m(x - p)^{m-1} q(x) + (x - p)^m q'(x)
\]
and \( f'(p) = 0 \). Also
\[
 f''(x) = m(m-1)(x - p)^{m-2} q(x) + 2m(x - p)^{m-1} q'(x) + (x - p)^m q''(x)
\]
and \( f''(p) = 0 \).
In general, for \( k \leq m \),
\[
 f^{(k)}(x) = \sum_{j=0}^{k} \binom{k}{j} \frac{d^j}{dx^j} (x - p)^m q^{(k-j)}(x)
\]
\[
 = \sum_{j=0}^{k} \binom{k}{j} m(m-1) \cdots (m-j+1)(x - p)^{m-j} q^{(k-j)}(x).
\]
Thus, for \( 0 \leq k \leq m - 1 \), we have \( f^{(k)}(p) = 0 \), but
\[
 f^{(m)}(p) = m! \lim_{x \to p} q(x) \neq 0.
\]
Conversely, suppose that \( f(p) = f'(p) = \ldots = f^{(m-1)}(p) = 0 \) and \( f^{(m)}(p) \neq 0 \). Consider the \((m-1)\)th Taylor polynomial of \( f \) expanded about \( p \):
\[
 f(x) = f(p) + f'(p)(x - p) + \ldots + \frac{f^{(m-1)}(p)(x - p)^{m-1}}{(m-1)!} + \frac{f^{(m)}(\xi(x))(x - p)^m}{m!}
\]
\[
 = (x - p)^m \frac{f^{(m)}(\xi(x))}{m!},
\]
where \( \xi(x) \) is between \( x \) and \( p \). Since \( f^{(m)} \) is continuous, let
\[
 q(x) = \frac{f^{(m)}(\xi(x))}{m!}.
\]
Then \( f(x) = (x - p)^m q(x) \) and
\[
 \lim_{x \to p} q(x) = \frac{f^{(m)}(p)}{m!} \neq 0.
\]
So \( p \) is a zero of multiplicity \( m \).

**Question 2.5.2:**

Apply Newton’s method to approximate a root of
\[
 f(x) = e^{6x} + 3(\ln 2)^2 e^{2x} - \ln 8e^{4x} - (\ln 2)^3 = 0.
\]
Generate terms until \( |p_{n+1} - p_n| < 0.0002 \), and construct the Aitken’s \( \Delta^2 \) sequence \( \{p_n\} \).

**SOLUTION:** Applying Newton’s method with \( p_0 = 0 \) requires finding \( p_{10} = -0.182888 \). For the Aitken’s \( \Delta^2 \) sequence, we have sufficient accuracy with \( p_6 = -0.183387 \). Newton’s method fails to converge quadratically because there is a multiple root.
Question 2.5.13:

2.5.13: (3 (a))

\[ P_n = \frac{1}{n}, \quad n \geq 1 \]

\[ P_n = P_n - \frac{1}{\Delta^2} \frac{(P_{n+1} - P_n)^2}{(P_{n+2} - 2P_{n+1} + P_n)} = \frac{1}{n} - \frac{(\frac{1}{n+1} - \frac{1}{n})^2}{(\frac{1}{n+2} - \frac{2}{n+1} + \frac{1}{n})} \]

\[ P_n \text{ conv. faster than } P_n. \]

Question 2.5.15:

2.5.15: \[ \lim_{n \to \infty} \frac{|P_{n+1} - P_n|}{|P_n - P_{n-1}|} = \lim_{n \to \infty} \frac{|(P_n - P_{n-1}) - (P_{n-1} - P_{n-2})|}{|P_n - P_{n-1}|} \]

\[ \leq \lim_{n \to \infty} \frac{|P_n - P_{n-1}| + |P_{n-1} - P_{n-2}|}{|P_n - P_{n-1}|} = 1 + \alpha \]

\[ \leq \lim_{n \to \infty} \frac{|P_{n+1} - P_n| - |P_{n-1} - P_{n-2}|}{|P_{n+1} - P_{n}|} \leq 1 - \alpha \]

Thus \( 1 - \alpha \leq \lim_{n \to \infty} \left| \frac{P_{n+1} - P_n}{P_n - P_{n-1}} \right| \leq 1 + \alpha \)

\[ \therefore \alpha \to 0 \text{ as } n \to \infty \]

Thus: \[ \lim_{n \to \infty} \left| \frac{P_{n+1} - P_n}{P_n - P_{n-1}} \right| = 1 \]
Question 2.5.14:

a. Show that a sequence \( \{p_n\} \) that converges to \( p \) with order \( \alpha > 1 \) converges superlinearly to \( p \).

b. Show that the sequence \( p_n = \frac{1}{n^n} \) converges superlinearly to 0, but does not converge of order \( \alpha \) for any \( \alpha > 1 \).

SOLUTION: Since \( \{p_n\} \) converges to \( p \) with order \( \alpha > 1 \), a positive constant \( \lambda \) exists with

\[
\lambda = \lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}}
\]

Hence

\[
\lim_{n \to \infty} \frac{|p_{n+1} - p|}{p_n - p} = \lim_{n \to \infty} \left( \frac{p_{n+1} - p}{p_n - p} \right)^{\alpha} = \lim_{n \to \infty} \frac{|p_n - p|^{\alpha-1}}{|p_n - p|^{\alpha}} = \lambda \cdot 0 = 0
\]

and

\[
\lim_{n \to \infty} \frac{p_{n+1} - p}{p_n} = 0.
\]

This implies that \( \{p_n\} \) that converges superlinearly to \( p \).

b. The sequence converges \( p_n = \frac{1}{n^n} \) superlinearly to zero because

\[
\lim_{n \to \infty} \frac{1/(n + 1)^{(n+1)}}{1/n^n} = \lim_{n \to \infty} \frac{n^n}{(n + 1)^{(n+1)}}
\]

\[
= \lim_{n \to \infty} \left( \frac{n}{n + 1} \right)^n \frac{1}{n + 1}
\]

\[
= \lim_{n \to \infty} \left( \frac{1}{(1 + 1/n)^n} \right) \frac{1}{n + 1} = \frac{1}{e} \cdot 0 = 0.
\]

However, for \( \alpha > 1 \), we have

\[
\lim_{n \to \infty} \frac{1/(n + 1)^{(n+1)}}{(1/n^n)^{\alpha}} = \lim_{n \to \infty} \frac{n^{\alpha n}}{(n + 1)^{(n+1)}}
\]

\[
= \lim_{n \to \infty} \left( \frac{n}{n + 1} \right)^n \frac{n^{(\alpha-1)n}}{n + 1}
\]

\[
= \lim_{n \to \infty} \left( \frac{1}{(1 + 1/n)^n} \right) \frac{n^{(\alpha-1)n}}{n + 1} = \frac{1}{e} \cdot \infty = \infty.
\]

So the sequence does not converge of order \( \alpha \) for any \( \alpha > 1 \).