Question 2.6.2(b):

Try Matlab Code below, you will find what happen :D

```matlab
function haha

x=0.28; %set starting value
nmax=25; %set max number of iterations
eps=1; %initialize error bound eps
xvals=x; %initialize array of iterates
n=0; %initialize n (counts iterations)

while eps>=1e-5 & n<=nmax %set while-conditions
    y=x-(10*x^3 - 8.3*x^2 + 2.295*x - 0.21141)/(30*x^2 - 16.6*x + 2.295); %compute next iterate
    xvals=[xvals;y]; %write next iterate in array
    eps=abs(y-x); %compute error
    x=y; n=n+1; %update x and n
end

x
```
Question 2.6.7:

SOLUTION:

a. Bissection method: For \( p_0 = 0.1 \) and \( p_1 = 1 \), we have \( p_{14} = 0.23233 \).

b. Newton’s method: For \( p_0 = 0.55 \), we have \( p_6 = 0.23235 \).

c. Secant method: For \( p_0 = 0.1 \) and \( p_1 = 1 \), we have \( p_8 = 0.23235 \).

d. Method of False Position: For \( p_0 = 0.1 \) and \( p_1 = 1 \), we have \( p_8 = 0.23025 \).

e. Müller’s method: For \( p_0 = 0 \), \( p_1 = 0.25 \), and \( p_2 = 1 \), we have \( p_6 = 0.23235 \).

Notice that the method of False Position for this problem was considerably less effective than both the Secant method and the Bisection method.

Question 3.1.2(a):

2. a. For the given functions \( f(x) = \sin xx \), let \( x_0 = 1, x_1 = 1.25 \), and \( x_2 = 1.6 \). Construct interpolation polynomials of degree at most one and at most two to approximate \( f(1.4) \), and find the absolute error.

SOLUTION: We have

\[
P_1(x) = \frac{(x - 1.6)}{(1.25 - 1.6)} f(1.25) + \frac{(x - 1.25)}{(1.6 - 1.25)} f(1.6) = -0.6969992408x + 0.164142691,
\]

with \( P_1(1.4) = -0.811656668 \) and \( |f(1.4) - P_1(1.4)| = 0.130399849 \);

\[
P_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)
\]

\[
= 3.552379809x^2 - 10.82128170x + 7.2089018882
\]

with \( P_2(1.4) = -0.9182280623 \) and \( |f(1.4) - P_2(1.4)| = 0.328284543 \times 10^{-1} \).

Question 3.1.4(a):

4. a. Use Theorem 3.3 to find an error bound for the approximations in Exercise 2(a).

SOLUTION: For the linear polynomial we use \( x_1 = 1.25 \) and \( x_2 = 1.6 \).

\[
|f(1.4) - P_1(1.4)| = \frac{|f'(\xi)|}{2!} |x - x_1||x - x_2| = 0.5 - \pi^2 \sin(\pi \xi) |1.4 - 1.25||1.4 - 1.6|
\]

\[
\leq 0.5\pi^2(0.15)(0.2) \leq 0.14804407.
\]

For the second degree polynomial we use \( x_1 = 1.25, x_2 = 1.6, \) and \( x_0 = 1.0 \).

\[
|f(1.4) - P_2(1.4)| = \frac{|f^{(3)}(\xi)|}{3!} |x - x_1||x - x_2||x_0 - x|
\]

\[
= \frac{1}{6} | - \pi^3 \cos(\pi \xi) |1.4 - 1.25||1.4 - 1.6||1.4 - 1.0| \leq 0.062012553.
\]
5. a. Use appropriate Lagrange interpolating polynomials of degrees one, two, and three to approximate
\( f(8.4) \) if \( f(8.1) = 16.94410, f(8.3) = 17.56492, f(8.6) = 18.50515, \) and \( f(8.7) = 18.82091 \)

SOLUTION: With \( x_0 = 8.1, y_0 = 16.94410, x_1 = 8.3, y_1 = 17.56492, x_2 = 8.6, y_2 = 18.50515, \)
\( x_3 = 8.73 \) and \( y_3 = 18.82091 \) we have

\[ P_1(x) = 3.1341x - 8.44811, \quad P_2(x) = 0.06x^2 + 2.1201x - 4.46531 \]

and

\[ P_3(x) = -0.00208333334x^3 + 0.1120833334x^2 + 1.686204167x - 2.96077250 \]

These interpolation polynomials give the following results.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_0, x_1, \ldots, x_n )</th>
<th>( P_3(8.4) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8.3, 8.6</td>
<td>17.87833</td>
</tr>
<tr>
<td>2</td>
<td>8.3, 8.6, 8.7</td>
<td>17.87716</td>
</tr>
<tr>
<td>3</td>
<td>8.3, 8.6, 8.7, 8.1</td>
<td>17.87714</td>
</tr>
</tbody>
</table>

Question 3.1.9:

9. Let \( P_3(x) \) be the interpolating polynomial for the data \( (0, 0), (0.5, y), (1, 3), \) and \( (2, 2) \). Find \( y \) if the
coefficient of \( x^3 \) in \( P_3(x) \) is 6.

SOLUTION: Solving for \( P_3(x) \) gives

\[ P_3(x) = \frac{(x - 0.5)(x - 1)(x - 2)}{(0 - 0.5)(0 - 1)(0 - 2)} \cdot 0 + \frac{(x - 0)(x - 1)(x - 2)}{(0.5 - 0)(0.5 - 1)(0.5 - 2)} \cdot y + \frac{(x - 0)(x - 0.5)(x - 2)}{(1 - 0)(1 - 0.5)(1 - 2)} \]

\[ + \frac{(x - 0)(x - 0.5)(x - 1)}{(2 - 0)(2 - 0.5)(2 - 1)} \cdot 2 \]

\[ = \frac{8}{3}x(x - 1)(x - 2)y - 6x(x - 0.5)(x - 2) + \frac{2}{3}x(x - 0.5)(x - 1) \]

\[ = \left( \frac{8}{3}y - 6 + \frac{2}{3} \right)x^3 + \ldots. \]

Since the coefficient of \( x^3 \) is given to be 6, we have \( 6 = \frac{8}{3}y - 6 + \frac{2}{3} \), so \( \frac{34}{3} = \frac{8}{3}y \) and \( y = 4.25 \).
Question 3.1.12:

12. Use the Lagrange polynomial of degree three and four-digit chopping arithmetic to approximate \cos 0.750, based on the following data:

\[
\cos 0.698 = 0.7661, \quad \cos 0.768 = 0.7193, \quad \cos 0.733 = 0.7432, \quad \cos 0.803 = 0.6946.
\]

SOLUTION: The four-digit chopping calculations for the Lagrange polynomial of degree three at 0.750 are

\[
P_3(0.750) = \left( \begin{array}{c}
0.750 - 0.733 \\
0.750 - 0.768 \\
0.750 - 0.803 \\
0.698 - 0.733 \\
0.698 - 0.768 \\
0.698 - 0.803
\end{array} \right) \left( \begin{array}{c}
0.750 - 0.733 \\
0.750 - 0.768 \\
0.750 - 0.803 \\
0.698 - 0.733 \\
0.698 - 0.768 \\
0.698 - 0.803
\end{array} \right)^{-1}
\]

\[
= \left( \begin{array}{c}
0.0170 \\
0.0350 \\
0.0520 \\
-0.0350 \\
-0.0520 \\
-0.0700
\end{array} \right) \left( \begin{array}{c}
0.7661 \\
0.0520 \\
0.0520 \\
-0.0350 \\
-0.0520 \\
0.0700
\end{array} \right)^{-1}
\]

\[
= \left( \begin{array}{c}
0.00001621 \\
0.00002572 \\
0.00000499 \\
0.00003890 \\
0.00000398 \\
-0.00002572
\end{array} \right)
\]

The actual error is 0.0004, and an error bound is \(2.7 \times 10^{-8}\). The discrepancy between the error bound and the actual error is due to the fact that the data is accurate to only four decimal places, significant round-off error occurs in the computation of the approximation.

Question 3.1.21:

22. Show that \(\max_{x_j \leq x \leq x_{j+1}} |g(x)| = \frac{h^2}{4}\), where \(g(x) = (x - jh)(x - (j + 1)h)\).

SOLUTION: Since \(g'(x) = 2x - 2jh - (j + 1)h\), we have \(g'(x) = 0\) if and only if \(x = (j + \frac{1}{2})h\). The maximum of \(g\) can occur only at an endpoint or a critical point, so

\[
\max |g(x)| = \max \left\{ |g(jh)|, \left| g\left((j + \frac{1}{2})h\right)\right|, |g((j + 1)h)| \right\} = \max \left\{ 0, \frac{h^2}{4}, \frac{h^2}{4} \right\} = \frac{h^2}{4}.
\]

Question 3.2.2(b):

2. b. Use Neville’s method to obtain the approximations for Lagrange interpolating polynomials of degree one, two, and three to approximate \(f(0)\) if \(f(-0.5) = 1.93750, f(-0.25) = 1.33203, f(0.25) = 0.800781\), and \(f(0.5) = 0.687500\).

SOLUTION: Neville’s method gives the table

<table>
<thead>
<tr>
<th>(x_i)</th>
<th>(P_0)</th>
<th>(P_1)</th>
<th>(P_2)</th>
<th>(P_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.5</td>
<td>1.93750</td>
<td>0.72656</td>
<td>0.95312</td>
<td>0.98437</td>
</tr>
<tr>
<td>-0.25</td>
<td>1.33203</td>
<td>1.06641</td>
<td>1.01562</td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>0.800781</td>
<td>1.01562</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.687500</td>
<td>0.91406</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The interpolation polynomials are

\[ P_1(x) = -1.06250x + 1.06641 \text{ with } P_1(0.0) = 1.06641, \]
\[ P_2(x) = 1.81250x^2 - 1.06250x + 0.95312 \text{ with } P_2(0.0) = 0.95312, \]
\[ P_3(x) = -1.00000x^3 + 1.31250x^2 - 1.00000x + 0.98437 \]

with \( P_3(0.0) = 0.98437 \), and

\[ |f(0.0) - P_1(0.0)| = 0.00641, \quad |f(0.0) - P_2(0.0)| = 0.0468760, \text{ and } |f(0.0) - P_3(0.0)| = 0.01563. \]

Question 3.2.3(ab):

3. a. Use Neville's method with the function \( f(x) = 3^x \) and the nodes \( x_0 = -2, x_1 = -1, x_2 = 0, \)
\( x_3 = 1, \text{ and } x_4 = 2 \) to approximate \( \sqrt{3} \).

b. Use Neville's method with the function \( f(x) = \sqrt{x} \) and the nodes \( x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 4, \)
\( x_4 = 5 \) to approximate \( \sqrt{3} \).

c. Which approximation is more accurate?

SOLUTION: a. The expected best approximation for this data is \( \sqrt{3} \approx P_4(1/2) \). To construct
\( P_4(1/2) \), we use the data in the following table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( P_0 )</th>
<th>( P_0,1 )</th>
<th>( P_{0,1,2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.1111111</td>
<td>0.6666667</td>
<td>1.5</td>
</tr>
<tr>
<td>-1</td>
<td>0.3333333</td>
<td>1.3333333</td>
<td>1.5</td>
</tr>
<tr>
<td>0</td>
<td>1.0000000</td>
<td>1.8333333</td>
<td>1.5</td>
</tr>
<tr>
<td>1</td>
<td>2.0000000</td>
<td>2.7777778</td>
<td>1.5</td>
</tr>
<tr>
<td>2</td>
<td>3.0000000</td>
<td>3.8125000</td>
<td>1.5</td>
</tr>
<tr>
<td>3</td>
<td>4.0000000</td>
<td>4.8437500</td>
<td>1.5</td>
</tr>
<tr>
<td>4</td>
<td>5.0000000</td>
<td>5.8666667</td>
<td>1.5</td>
</tr>
</tbody>
</table>

This gives
\[ \sqrt{3} \approx P_4(1/2) = P_{0,1,2,3,4} = 1.7083. \]

b. The expected best approximation for this data is \( \sqrt{3} \approx P_4(3) \). To construct
\( P_4(3) \), we use the data in the following table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( P_0 )</th>
<th>( P_{0,1} )</th>
<th>( P_{0,1,2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>5</td>
<td>3</td>
</tr>
</tbody>
</table>

This gives
\[ \sqrt{3} \approx P_4(3) = P_{0,3,2,3,4} = 1.690697. \]
Question 3.2.6:

Neville’s method is used to approximate \( f(0.5) \), giving the following table. Determine \( P_2 = f(0.7) \).

\[
\begin{array}{c|c|c|c|c}
& x_0 = 0 & P_0 = 0 & x_1 = 0.4 & P_1 = 2.8 & P_{01} = 3.5 \\
& x_2 = 0.7 & P_2 & & P_{12} & P_{012} = 27 \\
\end{array}
\]

SOLUTION: We use the formula

\[
P_{0,1,2} = \frac{(x-x_0)P_{1,2} - (x-x_2)P_{0,1}}{x_2-x_0}
\]

to obtain \( P_{1,2} \). After substitution we have

\[
\frac{27}{7} = \frac{(0.5 - 0)P_{1,2} - (0.5 - 0.7)3.5}{0.7 - 0} \quad \text{or} \quad \frac{27}{7} = \frac{0.5P_{1,2} + 0.7}{0.7}
\]

so \( P_{1,2} = 4 \). We then use the formula

\[
P_{1,2} = \frac{(x-x_1)P_2 - (x-x_2)P_1}{x_2-x_1}
\]

to obtain \( P_2 \). After substitution we have

\[
4 = \frac{(0.5 - 0.4)P_2 - (0.5 - 0.7)2.8}{0.3} \quad \text{and} \quad 1.2 = 0.1P_2 + 0.56,
\]

so \( P_2 = f(0.7) = 6.4 \).

Question 3.2.10:

10. Neville’s Algorithm is used to approximate \( f(0) \) given \( f(-2), f(-1), f(1), \) and \( f(2) \). Suppose \( f(-1) \) was overstated by 2 and \( f(1) \) was understated by 3. Determine the error in the original calculation of the value of the interpolating polynomial to approximate \( f(0) \).

SOLUTION: The errors created in Neville’s method are shown in the table. We are given \( e_1 = 2 \) and \( e_2 = -3 \) and assume that \( e_0 = e_3 = 0 \).

\[
\begin{array}{c|c|c|c|c|c|c}
& x_0 = -2 & e_0 = 0 & x_1 = -1 & e_1 = 2 & e_{0,1} = 4 \\
x_2 = 1 & e_2 = -3 & e_{1,2} = -\frac{1}{2} & e_{0,1,2} = 1 \\
x_3 = 2 & e_3 = 0 & e_{2,3} = -6 & e_{1,2,3} = -\frac{7}{3} & e_{0,1,2,3} = -\frac{2}{3}
\end{array}
\]

The original \( P_{0,1,2,3} \) is in error by the amount \( -\frac{2}{3} \).