Question 3.3.1(a):

1. a. Use Eq. (3.10) or Algorithm 3.2 to construct interpolating polynomials of degree one, two, and three for the following data to approximate $f(8.4)$ if $f(8.1) = 16.94410$, $f(8.3) = 17.56492$, $f(8.6) = 18.50515$, and $f(8.7) = 18.82091$.

SOLUTION: We have

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{(x_1 - x_0)} = 3.10410,$$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{(x_2 - x_1)} = 3.13410,$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{(x_2 - x_0)} = 0.06000,$$

$$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{(x_3 - x_2)} = 3.15760,$$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{(x_3 - x_1)} = 0.05875,$$

and

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{(x_3 - x_0)} = -0.002083.$$

So

$$P_1(x) = 16.94410 + 3.10410(x - x_0), \quad P_1(8.4) = 17.8753000;$$

$$P_2(x) = P_1(x) + 0.06000(x - x_0)(x - x_1), \quad P_2(8.4) = 17.8771300;$$

$$P_3(x) = P_2(x) - 0.002083(x - x_0)(x - x_1)(x - x_2), \quad P_3(8.4) = 17.87714250.$$

Question 3.3.4(a):

4. a. Use the Newton forward-difference formula to construct interpolating polynomials of degree one, two, and three for the following data to approximate $f(0.43)$ if $f(0) = 1$, $f(0.25) = 1.64872$, $f(0.5) = 2.71828$, and $f(0.75) = 4.48169$.

SOLUTION: The first divided differences are

$$\delta f(x_0) = f(x_1) - f(x_0) = 0.64872, \quad \delta f(x_1) = 1.06956, \quad \delta f(x_2) = 1.76341,$$

The second divided differences are

$$\delta^2 f(x_0) = \delta f(x_1) - \delta f(x_0) = 0.42084, \quad \delta^2 f(x_1) = 0.69385,$$

and the third divided difference is

$$\delta^3 f(x_0) = \delta^2 f(x_1) - \delta^2 f(x_0) = 0.27301.$$

In the following equations, we have $s = (1/h)(x - x_0)$.

$$P_1(s) = 1.0 + 0.64872s, \quad P_1(0.43) = 2.11580$$

$$P_2(s) = P_1(s) + 0.21042s(s - 1), \quad P_2(0.43) = 2.37638$$

$$P_3(s) = P_2(s) + 0.04550s(s - 1)(s - 2), \quad P_3(0.43) = 2.36060$$
Question 3.3.6(a):

6. a. Use the Newton back-difference formula to construct interpolating polynomials of degree one, two, and three for the following data to approximate \( f(0.43) \) if \( f(0) = 1, f(0.25) = 1.64872, f(0.5) = 2.71828, \) and \( f(0.75) = 4.48169. \)

SOLUTION: We have

\[
\nabla f(x_1) = f(x_1) - f(x_0) = 0.64872, \quad \nabla f(x_2) = 1.06956, \quad \nabla f(x_3) = 1.76341.
\]

The second divided differences are

\[
\nabla^2 f(x_2) = \nabla f(x_2) - \nabla f(x_1) = 0.42084, \quad \nabla^2 f(x_3) = 0.69385,
\]

and the third divided difference is

\[
\nabla^3 f(x_3) = \nabla^2 f(x_3) - \nabla^2 f(x_2) = 0.27301.
\]

In the following equations, we have \( x = 0.43, h = 0.25, \) and \( s = (1/h)(x - x_3) = -1.28. \)

\[
P_1(x) = f_3(x) - s \nabla f(x_3) = 2.22453, \\
P_2(x) = P_1(x) + (-1)^2 \frac{(-s(-s-1))}{2} \nabla^2 f(x_3) = 2.34886; \\
P_3(x) = P_2(x) + (-1)^3 \frac{(-s(-s-1)(-s-2))}{6} \nabla^3 f(x_3) = 2.36060.
\]

Question 3.3.7(a):

7. a. Use Algorithm 3.2 to construct the interpolating polynomial of degree three for the unequally spaced points given in the following table:

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.1</td>
<td>5.30000</td>
</tr>
<tr>
<td>0.0</td>
<td>2.00000</td>
</tr>
<tr>
<td>0.2</td>
<td>3.19000</td>
</tr>
<tr>
<td>0.3</td>
<td>1.00000</td>
</tr>
</tbody>
</table>

b. Add \( f(0.35) = 0.97260 \) to the table, and construct the interpolating polynomial of degree four.

SOLUTION:

a. Using Newton Divided-Difference Formula we have

\[
x_0 = -0.1, \quad x_1 = 0.0, \quad x_2 = 0.2, \quad x_3 = 0.3;
\]

\[
f[x_0] = 5.3, \quad f[x_1] = 2.0, \quad f[x_2] = 3.19, \quad f[x_3] = 1.0, \\
f[x_0,x_1] = 33.00000, \quad f[x_1,x_2] = 5.95000, \quad f[x_2,x_3] = -21.90000, \\
f[x_0,x_1,x_2] = 129.83333, \quad f[x_1,x_2,x_3] = -556.66667, \quad \text{and} \quad f[x_0,x_1,x_2,x_3] = -556.66667.
\]

This gives

\[
P_3(x) = 5.3 - 33(x + 0.1) + 129.83333(x + 0.1)x - 556.66667(x + 0.1)x(x - 0.2) \\
P_4(x) = P_3(x) + 2730.243387(x + 0.1)x(x - 0.2)(x - 0.3)
\]
Question 3.3.13:

The Newton forward divided-difference formula is used to approximate \( f(0.3) \) given the following data.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>15.0</td>
<td>21.0</td>
<td>30.0</td>
<td>51.0</td>
</tr>
</tbody>
</table>

Suppose it is discovered that \( f(0.4) \) was understated by 10 and \( f(0.6) \) was overstated by 5. By what amount should the approximation to \( f(0.3) \) be changed?

SOLUTION: The following is the incorrect divided-difference table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_0 = 0.0 )</td>
<td>( f[x_0] )</td>
</tr>
<tr>
<td>( x_1 = 0.2 )</td>
<td>( f[x_1] )</td>
</tr>
<tr>
<td>( x_2 = 0.4 )</td>
<td>( f[x_2] - 10 )</td>
</tr>
<tr>
<td>( x_3 = 0.6 )</td>
<td>( f[x_3] + 5 )</td>
</tr>
</tbody>
</table>

The incorrect polynomial \( Q(x) \) is given by

\[
Q(x) = f[x_0] + f[x_0, x_1](x - x_0) + (f[x_0, x_1, x_2] - 125)(x - x_0)(x - x_1) + (f[x_0, x_1, x_2, x_3] + 125)(x - x_0)(x - x_1)(x - x_2)
\]

Thus

\[
Q(0.3) = P_3(0.3) - 125(0.3)(0.1) + 729.1\bar{b}(0.3)(0.1)(-0.1) = P_3(0.3) - 5.9375.
\]

To obtain the approximation \( P_3(0.3) \) to \( f(0.3) \), we need to add 5.9375 to \( Q(0.3) \), the number we originally obtained.

Question 3.3.15:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( p(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(1) = 2e )</td>
<td></td>
</tr>
<tr>
<td>( P(2) = 3e )</td>
<td></td>
</tr>
<tr>
<td>( P(3) = 5e )</td>
<td></td>
</tr>
<tr>
<td>( P(1) = 27e )</td>
<td></td>
</tr>
<tr>
<td>( P(2) = 25e )</td>
<td></td>
</tr>
<tr>
<td>( P(3) = 22e )</td>
<td></td>
</tr>
</tbody>
</table>

Thus \( \Delta^2 P(10) = P(10) - 2P(11) + P(12) \)
7. For a function $f$, the forward divided-differences are given by the following table.

<table>
<thead>
<tr>
<th>$x_0 = 0$</th>
<th>$f[x_0]$</th>
<th>$f[x_0, x_1]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1 = 0.4$</td>
<td>$f[x_1]$</td>
<td>$f[x_0, x_1, x_2] = \frac{50}{7}$</td>
</tr>
<tr>
<td>$x_2 = 0.7$</td>
<td>$f[x_2] = 6$</td>
<td>$f[x_1, x_2] = 10$</td>
</tr>
</tbody>
</table>

Determine the missing entries in the table.

**SOLUTION:** We have the formula

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0},$$

and substituting gives

$$\frac{50}{7} = \frac{1}{0.7} (10 - f[x_0, x_1]). \quad \text{Thus} \quad f[x_0, x_1] = -0.7 \cdot \frac{50}{7} + 10 = 5.$$

Using the formula

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1},$$

and substituting gives

$$10 = \frac{1}{0.3} (6 - f[x_1]). \quad \text{Thus} \quad f[x_1] = 6 - 3 = 3.$$

Further,

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0},$$

so

$$5 = \frac{1}{0.4} (3 - f[x_0]) \quad \text{and} \quad f[x_0] = 1.$$
Question 3.3.22:

Show that

\[ f[x_0, x_1, \ldots, x_n, x] = f^{(n+1)}(\xi(x)) \frac{(x - x_0) \cdots (x - x_n)}{(n + 1)!}, \]

for some \( \xi(x) \) between \( x_0, x_1, \ldots, x_n, \) and \( x. \)

SOLUTION: Equation (3.3) gives

\[ f(x) = P_n(x) + \frac{f^{(n+1)}(\xi(x))}{(n + 1)!} (x - x_0) \cdots (x - x_n). \]

Let \( x_{n+1} = x. \) The interpolation polynomial of degree \( n + 1 \) on \( n + 2 \) nodes \( x_0, x_1, \ldots, x_{n+1} \) is

\[ P_{n+1}(t) = P_n(t) + f[x_0, x_1, \ldots, x_n, x_{n+1}] (t - x_0) (t - x_1) \cdots (t - x_n). \]

Since \( f(x) = P_{n+1}(x) \) and \( x = x_{n+1}, \) we have

\[ P_{n+1}(x) = P_n(x) + f[x_0, x_1, \ldots, x_n, x] (x - x_0) (x - x_1) \cdots (x - x_n) \]

and

\[ P_{n+1}(x) = P_n(x) + \frac{f^{(n+1)}(\xi(x))}{(n + 1)!} (x - x_0) \cdots (x - x_n). \]

Hence

\[ P_n(x) + \frac{f^{(n+1)}(\xi(x))}{(n + 1)!} (x - x_0) \cdots (x - x_n) = P_n(x) + f[x_0, \ldots, x_n, x] (x - x_0) \cdots (x - x_n), \]

which implies that

\[ f[x_0, \ldots, x_n, x] = \frac{f^{(n+1)}(\xi(x))}{(n + 1)!}. \]
Question 3.4.2(c):

2. c. Use Theorem 3.9 or Algorithm 3.3 to construct an approximating polynomial for the following data.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( f'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-0.29004996</td>
<td>-2.8019975</td>
</tr>
<tr>
<td>0.2</td>
<td>-0.56079734</td>
<td>-2.6150201</td>
</tr>
<tr>
<td>0.3</td>
<td>-0.81401972</td>
<td>-2.9734638</td>
</tr>
</tbody>
</table>

SOLUTION: We have

\[ z_0 = z_1 = 0.1, \quad z_2 = z_3 = 0.2, \quad z_4 = z_5 = 0.3, \]

and

\[ f[z_0] = f[z_1] = -0.29004996, \quad f[z_2] = f[z_3] = -0.56079734, \quad f[z_4] = f[z_5] = -0.81401972. \]

This gives

\[ f[z_0, z_1] = -2.8019975, \quad f[z_1, z_2] = -2.7074738, \quad f[z_2, z_3] = -2.6150201, \]
\[ f[z_3, z_4] = -2.5322238, \quad f[z_4, z_5] = -2.4533949, \]
\[ f[z_0, z_1, z_2] = 0.945237, \quad f[z_1, z_2, z_3] = 0.915537, \quad f[z_2, z_3, z_4] = 0.836963, \]
\[ f[z_3, z_4, z_5] = 0.788289. \]

Thus

\[ H_3(x) = -0.00985 - 2.802900x + 0.94524(x - 0.1)^2 - 0.29700(x - 0.1)^2(x - 0.2) \]
\[ -0.47935(x - 0.1)^2(x - 0.2)^2 + 0.05000(x - 0.1)^2(x - 0.2)^2(x - 0.3). \]

Question 3.4.4(c):

4. c. The data in Exercise 2(c) were generated using \( f(x) = x^2 \cos x - 3x \). Use the polynomials constructed in Exercise 2 for the given value of \( x \) to approximate \( f(0.18) \), and calculate the absolute error.

SOLUTION: For \( f(x) = x^2 \cos(x) - 3x \) and the polynomial \( H_5(x) \) in Exercise 2(c) we have, if we keep 10 decimal places,

\[ f(0.18) = -0.5081234644, \quad H_5(0.18) = -0.5081234698, \quad \text{and} \quad |f(0.18) - H_5(0.18)| = 5.4 \times 10^{-3}. \]

Question 3.4.6(a):

Apply Algorithm 3.3
SOLUTION: \( a. \) To show uniqueness, we assume that \( P(x) \) is another polynomial with \( P'(x_k) = f(x_k) \) and \( P'(x_k) = f'(x_k) \), for \( k = 0, \ldots, n \), and that the degree of \( P(x) \) is at most \( 2n + 1 \). Let
\[
D(x) = H_{2n+1}(x) - P(x).
\]
Then \( D(x) \) is a polynomial of degree at most \( 2n + 1 \) with \( D(x_k) = 0 \) and \( D'(x_k) = 0 \), for each \( k = 0, 1, \ldots, n \). Thus, \( D \) has zeros of multiplicity 2 at each \( x_k \), so
\[
D(x) = (x - x_0)^2 \cdots (x - x_n)^2 Q(x).
\]
Either, \( D(x) \) is of degree \( 2n \) or more, which would be a contradiction, or \( Q(x) \equiv 0 \), which implies that \( D(x) \equiv 0 \). This implies that \( P(x) \) is \( H_{2n+1}(x) \), so this polynomial is unique.

To show that the error assumes the form given, we first note that this error formula holds if \( x = x_k \) for any choice of \( \zeta \). When \( x \neq x_k \), for \( k = 0, \ldots, n \), we define
\[
g(t) = f(t) - H_{2n+1}(t) + \frac{(t - x_0)^2 \cdots (t - x_n)^2}{(x - x_0)^2 \cdots (x - x_n)^2} [f(x) - H_{2n+1}(x)].
\]
Note that \( g(x_k) = 0 \), for \( k = 0, \ldots, n \), and \( g(x) = 0 \). Thus, \( g \) has \( n + 2 \) distinct zeros in \( [a, b] \). By Rolle’s Theorem, \( g' \) has \( n + 1 \) distinct zeros, \( \xi_0, \ldots, \xi_n \), which are between the numbers \( x_0, \ldots, x_n, x \).

In addition, \( g'(x_k) = 0 \), for \( k = 0, \ldots, n \), so \( g' \) has \( 2n + 2 \) distinct zeros \( \xi_0, \ldots, \xi_n, x_0, \ldots, x_n \). Since \( g' \) is \( 2n + 1 \) times differentiable, the Generalized Rolle’s Theorem 1.10 implies that a number \( \zeta \) in \( [a, b] \) exists with \( g^{(2n+2)}(\zeta) = 0 \).

However,
\[
g^{(2n+2)}(t) = f^{(2n+2)}(t) - \frac{d^{2n+2} H_{2n+1}(t)}{dt^{2n+2}} - \frac{[f(x) - H_{2n+1}(x)]}{(x - x_0)^2 \cdots (x - x_n)^2} \frac{d^{2n+2} [(t - x_0)^2 \cdots (t - x_n)^2]}{dt^{2n+2}}.
\]

Since \( H_{2n+1}(t) \) is a polynomial of degree \( 2n + 1 \), its derivative of this order is zero. Also,
\[
\frac{d^{2n+2} [(t - x_0)^2 \cdots (t - x_n)^2]}{dt^{2n+2}} = (2n + 2)!.
\]

As a consequence, a number \( \zeta \) exists with
\[
0 = g^{(2n+2)}(\zeta) = f^{(2n+2)}(\zeta) - \frac{[f(x) - H_{2n+1}(x)]}{(x - x_0)^2 \cdots (x - x_n)^2} (2n + 2)!,
\]

so
\[
f(x) - H_{2n+1} = \frac{(x - x_0)^2 \cdots (x - x_n)^2}{(2n + 2)!} f^{(2n+2)}(\zeta),
\]

for some \( \zeta \) between \( x_0, x_1, \ldots, x_n \), and \( x \).