Math 171B Homework Assignment #6

Due date: June 1, 2018

1. (10 points) For \( f(x) = x_1^4 - x_1x_2 + 20x_2^2 \), apply the improved version of steepest descent method to find a minimizer with starting point \( x^0 = (1, 1) \). Find the explicit iteration formula for \( \{x^k\} \). Plot the points \( x^0, x^1, \cdots, x^{20} \) in the plane by Matlab. What is the limit of the sequence \( \{x^k\} \)?

2. (10 points) For the quadratic function \( f(x) = x_1^2 + 2x_2^2 + 4x_1 + 4x_2 \), apply the improved version of the steepest descent method to find a minimizer with the starting point \( x^1 = (0, 0) \). Use induction to show that the sequence \( x^k \) has the expression
\[
x^k = (2/3)^{k-1} - 2, (-1/3)^{k-1} - 1.
\]
Show that the sequence \( x^k \) converges to the unique minimizer of \( f(x) \).

3. (10 points) For the quartic function \( f(x) = x_1^4 + x_2^4 + x_1^2 - x_1x_2 + x_2^2 + x_1 - 2x_2 \), apply Newton’s method to find a minimizer with starting point \( x^0 = (0, 0) \). Find the explicit iteration formula for \( \{x^k\} \). Plot the points \( x^0, \cdots, x^5 \) in the plane by matlab. Is \( x^5 \) a good approximate minimizer? If so, give reasons for justification.

4. (10 points) Let \( f(x) \) be differentiable in \( \mathbb{R}^n \) and \( p_k \) be a descent direction at point \( x_k \), set
\[
q(x) = \nabla f(x_k)^T (x - x_k) + \frac{1}{2}(x - x_k)^T B_k (x - x_k).
\]
With \( B_k \) being symmetric positive definite. Show that
\[
\lim_{\alpha \to 0} \frac{f(x_k) - f(x_k + \alpha p_k)}{q(x_k) - q(x_k + \alpha p_k)} = 1.
\]

5. For a differentiable function \( f : \mathbb{R}^n \to \mathbb{R} \), let \( g(x) := \nabla f(x) \) be its gradient. Consider the optimization problem
\[
\min_{0 \neq p \in \mathbb{R}^n} \frac{g^T p}{||p||^2}
\]
show that \( p^* = -g \) is a solution to the optimization problem by Cauchy-Schwarz inequality. It shows why \( -\nabla f(x) \) is the steepest descent direction.

6. Let \( f(x) = 2x_1^2 + 2x_2^2 + 4x_1x_2 + x_1 + x_2 \). Use unmodified steepest descent method \( (x_{k+1} = x_k + p_k) \) and modified steepest descent method \( (x_{k+1} = x_k + \alpha_k p_k) \) to find minimizer of \( f(x) \) with starting point \( x_0 = (0, 0) \) respectively. Which one is better?

7. (10 points) A function \( G : \mathbb{R}^n \to \mathbb{R}^m \) is called Lipschitz continuous if for any \( x, y \in \mathbb{R}^n \),
\[
||G(x) - G(y)|| \leq M||x - y||
\]
for some constant \( M \). Consider \( F(x) : \mathbb{R}^n \to \mathbb{R} \), if \( F'(x) \) is Lipschitz continuous with Lipschitz constant \( M \), prove that the error in the linear model \( L_k(x) = F(x^k) + F'(x^k)(x - x^k) \) of \( F(x) \) can be bounded as
\[
||F(x) - L_k(x)|| \leq \frac{1}{2}M||x - x^k||^2.
\]