1. (10 points) Determine whether or not the set $\mathcal{E} = \{ x \in \mathbb{R}^2 : 2x_1^2 + 3x_2^2 + 4x_1x_2 \leq 1 \}$ is convex. Give reasons.

Sol: Claim: If $A$ is positive semidefinite, then $\varepsilon = \{ x \mid x^T Ax \leq 1 \}$ is convex.

It is easy to check above matrix is positive definite, so $\mathcal{E}$ is convex.

Here is proof of claim:

$A$ is symmetric positive semidefinite. By Cholesky decomposition, there exists matrix $R$, such that $A = R^T R$. By definition, to show set $\varepsilon$ is convex, we need to prove, for any $x, y \in \varepsilon$, and $\forall \alpha \in [0, 1]$, we have

$$\alpha x + (1 - \alpha) y \in \varepsilon.$$ 

As $x, y \in \varepsilon$, then $x^T Ax = \| Rx \|_2^2 \leq 1$, and $y^T Ay = \| Ry \|_2^2 \leq 1$. Let

$$z = \alpha x + (1 - \alpha) y$$

$$z^T Az = \| Rz \|_2^2 = \| \alpha Rx + (1 - \alpha) Ry \|_2^2 \leq (\alpha \| Rx \|_2 + (1 - \alpha) \| Ry \|_2)^2$$

So $z^T Az \leq (\alpha + 1 - \alpha)^2 = 1$ and $z \in \varepsilon$, which proves set $\varepsilon$ is convex.

2. (10 points) Determine the convergence order of the sequence $\{ k \sin(\frac{1}{k}) \}_{k=1}^{\infty}$.

Sol: We know $\lim_{k \to \infty} k \sin(1/k) = 1$

$$\lim_{k \to \infty} \frac{(k + 1) \sin(\frac{1}{k+1}) - 1}{k \sin(1/k) - 1} = \lim_{k \to \infty} \frac{\sin(\frac{1}{k+1}) - \frac{\cos(\frac{1}{k+1})}{k+1}}{\sin(\frac{1}{k}) - \frac{\cos(\frac{1}{k})}{k}} = \lim_{n \to \infty} \frac{\frac{-\sin(\frac{1}{n+1})}{(k+1)^2}}{-\frac{\sin(\frac{1}{k})}{k^3}} = 1$$

Thus it has convergence order 1.

3. (10 points) Find the biggest integer $r > 0$ such that

$$\left( 1 + \frac{1}{n} \right)^n = e + O\left( \frac{1}{n^r} \right).$$
Sol:
\[
\lim_{n \to \infty} \left( \frac{1 + \frac{1}{n}}{1/n^r} \right) = \lim_{n \to \infty} \frac{(\frac{1}{n} + 1)^n}{n^r} \left( \log (\frac{1}{n} + 1) - \frac{1}{(\frac{1}{n} + 1)^n} \right) = \lim_{n \to \infty} \frac{e \left( \log (\frac{1}{n} + 1) - \frac{1}{(\frac{1}{n} + 1)^n} \right)}{n^{1-r}}
\]
\[
\lim_{n \to \infty} \frac{\log (\frac{1}{n} + 1) - \frac{1}{(\frac{1}{n} + 1)^n}}{n^{1-r}} = \lim_{n \to \infty} -e \frac{(\log (\frac{1}{n} + 1) - \frac{1}{(\frac{1}{n} + 1)^n})}{n^{1-r}}
\]
So the limit is finite if and only if \( r = 1 \). The largest \( r \) is 1.

4. (10 points) Find the gradient and hessian of the function
\[
f(x) = \frac{x_1^2x_2^2}{x_1^3 + x_2^3}, \text{ where } x = (x_1, x_2).
\]
Solution: Gradient
\[
\nabla f(x) = \begin{bmatrix}
\frac{\partial f(x)}{\partial x_1} \\
\frac{\partial f(x)}{\partial x_2}
\end{bmatrix} = \begin{bmatrix}
\frac{2x_1x_2(x_1^2 - x_2^2)}{(x_1^3 + x_2^3)^2} \\
\frac{2x_1x_2(x_2^3 - x_1^3)}{(x_1^3 + x_2^3)^2}
\end{bmatrix}
\]
Hessian
\[
\nabla^2 f(x) = \begin{bmatrix}
\frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} \\
\frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \frac{\partial^2 f(x)}{\partial x_2^2}
\end{bmatrix} = \begin{bmatrix}
\frac{2x_1^2(x_1^3 - 6x_1^2x_2^2 + x_2^4)}{(x_1^3 + x_2^3)^3} & -\frac{4x_1x_2(x_1^3 - 6x_1^2x_2^2 + x_2^4)}{(x_1^3 + x_2^3)^3} \\
-\frac{4x_1x_2(x_1^3 - 6x_1^2x_2^2 + x_2^4)}{(x_1^3 + x_2^3)^3} & \frac{2x_1^2(x_1^3 - 6x_1^2x_2^2 + x_2^4)}{(x_1^3 + x_2^3)^3}
\end{bmatrix}
\]

5. (10 points) Determine whether the following function
\[
f(x) := x_1^4 + x_2^4 + 4x_1^2x_2^2
\]
is convex or not.
Sol:
\[
\nabla^2 f(x) = \begin{bmatrix}
12x_1^2 + 8x_2^2 & 16x_1x_2 \\
16x_1x_2 & 12x_2^2 + 8x_1^2
\end{bmatrix}
\]
By calculation the determinant of Hermitian matrix is
\[
24(3x_1^2 + 3x_2^2 + (x_1 - x_2)^2) \geq 0
\]
And diagonal entries nonnegative, so the Hermitian is positive semidefinite and thus \( f(x) \) is convex.
6. (10 points) Suppose \( f(x) \in C[a, b] \). Show that there exists a number \( \xi \in [a, b] \) such that

\[
f(\xi) = \frac{4f(a) + 3f(b)}{7}.
\]

Solution: Let \( \alpha = \frac{4}{7} \in [0, 1] \), then

\[
c = \alpha f(a) + (1 - \alpha) f(b) \in [\min\{f(a), f(b)\}, \max\{f(a), f(b)\}]
\]

By Intermediate value theorem, there exists \( \xi \in [a, b] \), such that \( f(\xi) = c \).

7. (10 points) Let \( a \in \mathbb{R}^n \), \( H \in \mathbb{R}^{n \times n} \) be a symmetric matrix, \( A \in \mathbb{R}^{n \times n} \), and \( f(x), g(x) \) be the functions on \( \mathbb{R}^n \) defined as

\[
f(x) = a^T x + \frac{1}{2} x^T H x, \quad g(x) = Ax.
\]

Find the gradient and Hessian of the composition \( h(x) = f(g(x)) \).

Solution: \( h(x) = f(g(x)) = a^T Ax + \frac{1}{2} x^T A^T H A x \), which is still a quadratic function. \( H \) is symmetric.

Gradient \( \nabla h(x) = A^T H A x + A^T a \),

Hessian \( \nabla^2 h(x) = A^T H A \).

Method 2: Use chain rule.

Let \( g : \mathbb{R}^k \rightarrow \mathbb{R}^m \) and \( f : \mathbb{R}^m \rightarrow \mathbb{R}^s \), and \( h \) is their composition, i.e.:

\[
h(x) = f(g(x)) : \mathbb{R}^k \rightarrow \mathbb{R}^s,
\]

then \( \nabla h(x) = \nabla g(x) \nabla f(g(x)) \).

In this question, \( k = m = n \), \( s = 1 \). So use the Chain rule:

\[
\nabla h(x) = \nabla g(x) \nabla f(g(x)).
\]

Review: For \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \), \( f(x) = [f_1(x), \cdots, f_m(x)]^T \), then

\[
\nabla f(x) = [\nabla f_1(x), \cdots, \nabla f_m(x)].
\]

By simple calculation, \( \nabla g(x) = A^T \), and \( \nabla f(g(x)) = a + H A x \), so we have:

\[
\nabla f(g(x)) = A^T (a + H A x) = A^T a + A^T H A x,
\]

which is the same as above. And Hessian is:

\[
\nabla^2 h(x) = \nabla (\nabla f(g(x))) = \nabla (A^T a + A^T H A x) = A^T H A.
\]
8. (10 points) Let $F : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. Let $x \in \mathbb{R}^n$ and 
$\phi(\xi) = F(x + \xi h)$ for a vector $h \in \mathbb{R}^n$. Use the Fundamental Theorem of Calculus to show that

$$F(x + h) = F(x) + \int_0^1 [F'(x + \xi h) - F'(x)] h d\xi.$$ 

Sol: By chain rule

$$\frac{dF(x + \xi h)}{d\xi} = F'(x + \xi h)h$$

Thus

$$\int_0^1 F'(x + \xi h)h d\xi = f(x + \xi h)|_0^1 = F(x + h) - F(x)$$

$$F(x) + F'(x)h + \int_0^1 [F'(x + \xi h) - F'(x)] h d\xi = F(x) + F'(x)h + F(x + h) - F(x) - F'(x)h = F(x + h)$$