1. (10 points) Find all the local minimizers of

$$\min x_1x_2 + x_2x_3 + x_3x_1$$

s.t. $$x_1^2 + x_2^2 + x_3^2 = 1.$$  

Which one of them is a global minimizer?

Solution: Write the objective as $$f(x) = x^TAx,$$ with the symmetric matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The optimality conditions imply that

$$2Ax = 2\lambda x, \quad x^Tx = 1.$$ 

So we can get that

$$f(x) = x^TAx = \lambda.$$ 

Note that

$$Ax = \lambda x, x^Tx = 1.$$ 

The Lagrange multiplier $$\lambda$$ is an eigenvalue of $$A,$$ $$x$$ is a normalized eigenvector, and the objective value equals $$\lambda.$$ By a calculation, the matrix $$A$$ has eigenvalues: $$-1, -1, 2.$$ The eigenvalue $$-1$$ has multiplicity 2. By a simple comparison, one can see that $$-1$$ is a local (also global) minimum value, and 2 is a local (also global) maximum value. A local (global) minimizer is given by the normalized eigenvector of $$A$$ with the eigenvalue $$-1,$$ for instance, they are

$$\frac{1}{\sqrt{2}}(1, -1, 0)$$

or

$$\frac{1}{\sqrt{6}}(1, 1, -2).$$

The local (global) minimizer is not unique for this optimization problem. Any normalized eigenvector of $$A$$ associated to the eigenvalue $$-1$$ is a local (global) minimizer.
2. (10 points) Let \((a, b) \neq (0, 0)\) be a real pair. Find a local maximizer of
\[
\max ax_1 + bx_2 \\
\text{s.t. } x_1^4 + x_2^4 = 1.
\]
Is it also a global maximizer?
Solution: Consider the following equivalent minimization problem:
\[
\min -ax_1 - bx_2 \\
\text{s.t. } x_1^4 + x_2^4 = 1.
\]
First order condition:
\[
\begin{bmatrix}
-a \\
-b
\end{bmatrix} = 4\lambda
\begin{bmatrix}
x_1^3 \\
x_2^3
\end{bmatrix}
\]
bring into \(x_1^4 + x_2^4 = 1\), we get two solutions:
\[
\lambda_1 = \frac{1}{4}(a^4 + b^4)^{\frac{3}{4}}, \quad x_1 = \left(\frac{-a}{4\lambda_1}\right)^{\frac{1}{4}}, x_2 = \left(\frac{-b}{4\lambda_1}\right)^{\frac{1}{4}}
\]
Or
\[
\lambda_2 = -\frac{1}{4}(a^4 + b^4)^{\frac{3}{4}}, \quad x_1 = \left(\frac{-a}{4\lambda_2}\right)^{\frac{1}{4}}, x_2 = \left(\frac{-b}{4\lambda_2}\right)^{\frac{1}{4}}
\]
check second order condition
\[
\nabla^2 f(x) - \lambda \nabla^2 c(x) = -\lambda
\begin{bmatrix}
12x_1^2 & 0 \\
0 & 12x_2^2
\end{bmatrix},
\]
which is positive semidefinite if \(\lambda \leq 0\), so the second solution is global minimizer, which is the global maximizer of the original problem.

3. (10 points) Find all the local minimizer of
\[
\min x_1^2 + x_2^2 + 4x_1x_2 \\
\text{s.t. } x_1^2 \leq 1, x_2^2 \leq 1.
\]
Identify the global minimizers of them.
Solution. By optimality condition, we need to decide which constraint is active.
There are three cases:
(1) first inequality constraint is active and second one is not active, then by first order optimality condition
\[
\nabla f(x) = \lambda \nabla c_1(x).
\]
If $c_1(x)$ is active, then $x_1^* = \pm 1$, so
\[
\begin{bmatrix}
2x_1 + 4x_2 \\
4x_1 + 2x_2
\end{bmatrix} = \lambda \begin{bmatrix}
-2x_1 \\
0
\end{bmatrix}.
\]
then $x_1^* = 1, x_2^* = -2, \lambda^* = 3$, and $x_1^* = -1, x_2^* = 2, \lambda^* = 3$, but $x_2^*$ is not feasible.

(2) second inequality constraint is active and the first one is not active, then by first order optimality condition
\[
\nabla f(x) = \lambda \nabla c_2(x).
\]
If $c_2(x)$ is active, then $x_2^* = \pm 1$, so
\[
\begin{bmatrix}
2x_1 + 4x_2 \\
4x_1 + 2x_2
\end{bmatrix} = \lambda \begin{bmatrix}
0 \\
-2x_2
\end{bmatrix}.
\]
then $x_1^* = -2, x_2^* = 1, \lambda^* = 3$, and $x_1^* = 2, x_2^* = -1, \lambda^* = 3$, the first inequality is not feasible.

(3) two inequalities are active, then there are four feasible points $x_1 = \pm 1, x_2 = \pm 1$.

At $(1,1)$, check
\[
\begin{bmatrix}
2x_1 + 4x_2 \\
4x_1 + 2x_2
\end{bmatrix} = \lambda_1 \begin{bmatrix}
-2x_1 \\
0
\end{bmatrix} + \lambda_2 \begin{bmatrix}
0 \\
-2x_2
\end{bmatrix},
\]
so first order condition is not satisfied, and $(1,1)$ is not minimizer.

At $(-1,-1)$,
\[
\begin{bmatrix}
-6 \\
-6
\end{bmatrix} = \lambda_1 \begin{bmatrix}
2 \\
0
\end{bmatrix} + \lambda_2 \begin{bmatrix}
0 \\
2
\end{bmatrix}, \quad \lambda_1 = -3, \quad \lambda_2 = -3.
\]
first order condition is not satisfied.,

At $(1,-1)$,
\[
\begin{bmatrix}
-2 \\
2
\end{bmatrix} = \lambda_1 \begin{bmatrix}
-2 \\
0
\end{bmatrix} + \lambda_2 \begin{bmatrix}
0 \\
2
\end{bmatrix}, \quad \lambda_1 = 1, \quad \lambda_2 = 1.
\]

At $(-1, 1)$,
\[
\begin{bmatrix}
2 \\
-2
\end{bmatrix} = \lambda_1 \begin{bmatrix}
2 \\
0
\end{bmatrix} + \lambda_2 \begin{bmatrix}
0 \\
-2
\end{bmatrix}, \quad \lambda_1 = 1, \quad \lambda_2 = 1.
\]

Check the second order condition for $(1,-1), (-1,1)$,
\[ \nabla^2 f(x) - \lambda_1 \nabla^2 c_1(x) - \lambda_2 \nabla^2 c_2(x) = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \succeq 0. \]

so for all \( p \in \text{Null}(J_a(x^*)) \), we have \( p^T H(x^*, \lambda^*) p \geq 0 \), second order condition is true, and we find two minimizers \((1, -1), (-1, 1)\), with minimum \( f(-1, 1) = f(1, -1) = -2\).

4. (10 points) Find the point on the parabola \( 5y = (x - 1)^2 \) that is closest to point \((1, 2)\). Formulate this problem as a constrained optimization, and then solve it by using optimality conditions.

Solution. The closest to point to \((1, 2)\), we formulate it as an optimization problem:

\[
\min (x - 1)^2 + (y - 2)^2 \\
\text{s.t.} (x - 1)^2 - 5y = 0.
\]

first order necessary condition \( \nabla f(x) = \lambda \nabla c(x) \), so

\[
\begin{bmatrix} 2(x - 1) \\ 2(y - 2) \end{bmatrix} = \lambda \begin{bmatrix} 2(x - 1) \\ -5 \end{bmatrix},
\]

With constraint \((x - 1)^2 = 5y\), we have \( x^* = 1, y^* = 0, \lambda = \frac{4}{5} \).

Second order condition

\[
\nabla^2 f(x) - \lambda \nabla^2 c(x) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \lambda^* \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \succeq 0.
\]

So for any \( p \in \text{Null}(\nabla c(x)) \), second order condition is satisfied. so \((1, 0)\) is the point on the parabola that is closest to point \((1, 2)\).

5. (10 points) Let \( A \in \mathbb{R}^{n \times n} \) be symmetric. If \( \lambda \) is a Lagrange multiplier of

\[
\begin{align*}
\min & \ x^T Ax \\
\text{s.t.} & \ x^T x - 1 = 0,
\end{align*}
\]

show that \( \lambda \) is an eigenvalue of \( A \).

Proof. \( A \) is symmetric, and \( \lambda \) is a Lagrange multiplier, by optimality condition, we have \( \nabla f(x) = \lambda \nabla c(x) \), so

\[ Ax = \lambda x. \]

So \( \lambda \) is an eigenvalue of \( A \). \( \square \)
6. (10 points) Let \( x^* \) be a local minimizer of
\[
\min g^T x + x^T H x \\
\text{s.t. } 1 - x^T x \geq 0,
\]
where \( g \in \mathbb{R}^n \) and \( H \in \mathbb{R}^{n \times n} \) is symmetric. Suppose \( \lambda^* \) is its Lagrange multiplier. If \( H + \lambda^* I \succeq 0 \), show that \( x^* \) is also a global minimizer.

Solution: \( \lambda^* \) is its Lagrange multiplier, \( \lambda^* \geq 0 \).

first order condition
\[
2Hx^* + g = -2\lambda^* x^*, \quad \text{so } g = -2\lambda^* x^* - 2Hx^*, \quad \text{and } \lambda^*(1 - (x^*)^T x^*) = 0
\]
\[
f(x) = x^T H x + g^T x = (x^* - (x^* - x))^T H (x^* - (x^* - x)) + g^T (x^* - (x^* - x)).
\]
Let \( p = x^* - x \),
\[
f(x) = (x^* - p)H(x^* - p) + g^T (x^* - p) = f(x^*) - 2p^T H x^* + p^T H p - g^T p.
\]
So \( f(x) - f(x^*) = -2p^T H x^* + p^T H p - g^T p = p^T (H + \lambda^* I_n) p - 2p^T H x^* - \lambda^* p^T p - g^T p \)

By above \( g \), we have \( g^T p = -2\lambda^* (x^*)^T p - 2p^T H x^* \),
\[
f(x) - f(x^*) = p^T (H + \lambda^* I_n) p - 2p^T H x^* - \lambda^* p^T p + 2\lambda^* (x^*)^T p + 2p^T H x^* \\
= p^T (H + \lambda^* I_n) p - \lambda^* p^T p + 2\lambda^* (x^*)^T p
\]
\[
-\lambda^* p^T p + 2\lambda^* (x^*)^T p = -\lambda^* [p^T p - 2(x^*)^T p + (x^*)^T x^*] + \lambda^* (x^*)^T x^* \\
= -\lambda^* (p - x^*)^T (p - x^*) + \lambda^* (x^*)^T x^* \\
= -\lambda^* x^T x + \lambda^* (x^*)^T x^*
\]
As \( \lambda^*(1 - (x^*)^T x^*) = 0 \), so \( \lambda^* = \lambda^* (x^*)^T x^* \), bring into above equation, we have
\[
f(x) - f(x^*) = p^T (H + \lambda^* I_n) p - \lambda^* p^T p + 2\lambda^* (x^*)^T p \\
= p^T (H + \lambda^* I_n) p - \lambda^* x^T x + \lambda^* \\
= p^T (H + \lambda^* I_n) p + \lambda^* (1 - x^T x)
\]
\( H + \lambda^* I \succeq 0 \), and \( x \) is feasible, so \( 1 - x^T x \geq 0 \), \( \lambda^* \geq 0 \).

So for any \( x \in X \), \( f(x) - f(x^*) \geq 0 \), which proves \( x^* \) is global minimizer.

7. (10 points) Let \( f(x), g(x) \) be two continuous functions. Consider the optimization problem
\[
\min f(x) \\
\text{s.t. } g(x) \geq 0
\]
Show that if \( x^* \) is a local minimizer and \( g(x^*) > 0 \), then \( x^* \) is an unconstrained local minimizer of \( f(x) \). Furthermore, if \( x^* \) is a local minimizer but it is not an unconstrained local minimizer of \( f(x) \), then \( g(x^*) = 0 \).
Proof. Let $F$ be the feasible region. $x^*$ is a local minimizer, then we can find a ball $B(x^*, r_1)$ such that

$$f(x) \geq f(x^*), \forall x \in B(x^*, r_2)$$

$g(x^*) > 0$, so there exists $r_2$ such that $B(x^*, r_2) \in F$. Let $r = \min(r_1, r_2)$, then

$$f(x) \geq f(x^*), \forall x \in B(x^*, r)$$

Thus $x^*$ is an unconstrained local minimizer of $f(x)$.

Prove by contradiction, if $g(x^*) > 0$, then $x^*$ is an unconstrained local minimizer of $f(x)$ which is a contradiction.

8. (10 points) Consider the optimization problem

$$\min \quad x_1^3 + x_2^3 + x_1 - 4x_2$$
$$\text{s.t.} \quad x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0$$

Find all local and global minimizers.

Sol: If both constraints are inactive, then

$$\left(\frac{3x_1^2 + 1}{3x_2^2 - 4}\right) = \lambda_0 \left(\begin{array}{c} 1 \\ 1 \end{array} \right)$$

It has no solution.

If $x_1 \geq 0$ is the only active constraint, then

$$\left(\frac{3x_1^2 + 1}{3x_2^2 - 4}\right) = \lambda_0 \left(\begin{array}{c} 1 \\ 1 \end{array} \right) + \lambda_1 \left(\begin{array}{c} 1 \\ 0 \end{array} \right)$$

And we know $x_1 = 0, x_2 = 1$. Thus $\lambda_0 = -1, \lambda_1 = 2$. Check second order condition:

$$\nabla^2 f(x) - \lambda_0 \nabla^2 c_0(x) - \lambda_1 \nabla c_1(x) = \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}$$

It satisfies second order necessary condition.

When $x_2 \geq 0$ is the only active constraint, then $x_1 = 1, x_2 = 0$.

$$\left(\frac{3x_1^2 + 1}{3x_2^2 - 4}\right) = \lambda_0 \left(\begin{array}{c} 1 \\ 1 \end{array} \right) + \lambda_2 \left(\begin{array}{c} 0 \\ 1 \end{array} \right)$$

Thus $\lambda_0 = 4, \lambda_2 = -8$. It is not a KKT point since $\lambda_2 < 0$.

$x_1 \geq 0, x_2 \geq 0$ cannot be both active. As a result, $x = (0, 1)$ is the only local minimizer and thus global minimizer.