1. (10 points) For the symmetric matrix
\[
A = \begin{bmatrix}
0 & -1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 2
\end{bmatrix}
\]
find an orthogonal matrix \( Q \) such that \( Q^T AQ = D \) is diagonal. What are the eigenvalues?

2. (10 points) For the matrix
\[
A = \begin{bmatrix}
1 & -1 & -1 \\
-1 & 2 & 1 \\
-1 & 1 & 3
\end{bmatrix}
\]
determine whether or not \( A \) is psd (or pd)? If it is, find a square matrix \( L \) such that \( A = LL^T \). If it is not, explain reasons.

3. (10 points) Find the singular value decomposition for the matrix
\[
A = \begin{bmatrix}
0 & 2 \\
-3 & 0 \\
0 & 0
\end{bmatrix}
\]

4. (10 points) Among all vectors \( x := (x_1, x_2) \in \mathbb{R}^2 \) with \( \|x\|_1 = 1 \), find the vector \( x \) such that the norm \( \|x_1 - 2\|_2 \|x_2 - 2\|_2 \) is minimum.

5. (10 points) Find the range of real number \( t \) such that the angle between the vectors \((1, t)\) and \((1, 1)\) is between \(0^\circ\) and \(60^\circ\).

6. (10 points) For \( x := (x_1, x_2) \in \mathbb{R}^2 \), define the function
\[
\|x\| := \sqrt{x_1^2 - 2x_1x_2 + 2x_2^2}.
\]
Show that \( \|x\| \) is a norm function on \( \mathbb{R}^2 \) and give a formula for the dual norm \( \|x\|_* \).

7. (10 points) If \( A \in \mathbb{R}^{n \times n} \) is symmetric positive definite, show that its inverse \( A^{-1} \) is also positive definite. If \( B \in \mathbb{R}^{n \times n} \) is invertible, show that \( B^T B \) is positive definite.

8. (10 points) If \( A \in \mathbb{R}^{n \times n} \) is symmetric, show that its biggest singular value equals the maximum absolute value of its eigenvalues.

9. (10 points) For each \( B \in \mathbb{R}^{m \times n} \), show that its biggest singular value \( \sigma_1 \) can be expressed as
\[
\sigma_1 = \max_{0 \neq x \in \mathbb{R}^m} \frac{x^T By}{\|x\|_2 \|y\|_2}.
\]

10. (10 points) Let \( C \in \mathbb{R}^{n \times n} \) be symmetric positive definite. For all \( x, y \in \mathbb{R}^n \), show that
\[
|x^T Cy| \leq \sqrt{x^T C x} \sqrt{y^T C y}.
\]