§3.3. The Univariate TMP.

A measure \( \mu : \mathbb{B} \rightarrow \mathbb{R}^+ \) on the Borel sets of \( \mathbb{R} \) satisfies

\[
\mu \left( \bigcup_{i=1}^{n} T_i \right) = \sum_{i=1}^{n} \mu(T_i)
\]

if the \( T_i \) are disjoint.

1. If \( \text{supp}(\mu) \) is a singleton, then \( \mu \) is called \( \mathbb{R} \)-atomic.

2. If \( \text{supp}(\mu) \) has \( r \) points, then \( \mu \) is called \( \mathbb{R} \)-atomic.

3. If \( \text{supp}(\mu) \cap \mathbb{K} = \emptyset \), then \( \mu \) is finitely atomic.

The support of \( \mu \) is the smallest closed set \( S \) such that \( \mu(\mathbb{R}\setminus S) = 0 \).

- If \( \text{supp}(\mu) \) is an open set \( O \) in \( \mathbb{R} \), then \( \mu(O) = 0 \).
- If \( \text{supp}(\mu) \) is open, then \( \mu(\emptyset) = 0 \).
Consider an interval $I \subseteq (-\infty, +\infty)$. $d$ is a degree.

**Definition:** The moment cone of degree $d$ with the support set $I$ is the set $\mathcal{M}_{I}^{d}$ that is Borel

$$\mathcal{R}_{d}(I) = \{ y = \int_{I} x^{r} d\alpha : \text{supp} (\alpha) \subseteq I \}$$

**Theorem:** $\mathcal{R}_{d}(I) = \text{cone} \left( \left\{ [u]_{d} : u \in I \right\} \right)$

or equivalently,

$$\mathcal{R}_{d}(I) = \left\{ \left[ \sum_{i=1}^{N} \lambda_{i} \cdot [t_{i}]_{d} \right] : \lambda_{i} \geq 0, \quad t_{i} \in I \right\}$$
\[ y = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_d \end{pmatrix} \in \mathbb{R}^{d+1}, \]

Write \( d = 2k \) (even) or \( d = 2k + 1 \) (odd).

The moment matrix \( M_k[y] \) is:

\[
M_k[y] := \begin{pmatrix}
y_0 & y_1 & \cdots & y_k \\
y_1 & y_2 & \cdots & y_{k+1} \\
\vdots & \vdots & \ddots & \vdots \\
y_k & y_{k+1} & \cdots & y_{2k} \\
\end{pmatrix}
\]

This is a Hankel matrix.

\[
M_{k-1}[y] = \begin{pmatrix}
y_0 & y_1 & \cdots & y_{k-1} \\
y_1 & y_2 & \cdots & y_k \\
\vdots & \vdots & \ddots & \vdots \\
y_{k-1} & y_k & \cdots & y_{2k-2} \\
\end{pmatrix}
\]
What is the truncated moment problem (TMP)?
For a given \( y \in \mathbb{R}^{d+1} \), decide \( y \in \mathbb{R}^d \)?

Definition 3.3.1: Let \( d = 2k \) or \( d = 2k + 1 \):

1) If \( M_k[y] \) has rank \( k+1 \) (i.e., \( M_k[y] \) is nonsingular),
    then we define \( \text{rank}(y) = k+1 \).

2) If \( \text{rank} \ M_k[y] < k+1 \), then \( \text{rank}(y) \) is defined to be the smallest \( r \) such that

\[
\begin{bmatrix}
y_r \\
y_{r+1} \\
\vdots \\
y_{r+k}
\end{bmatrix} \in \text{Range} 
\begin{bmatrix}
y_0 & y_1 & \cdots & y_{r-1} \\
y_1 & y_2 & \cdots & y_r \\
\vdots & \vdots & \ddots & \vdots \\
y_k & y_{k+1} & \cdots & y_{r+k-1}
\end{bmatrix}.
\]
TMP: For given \( y = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_k \end{bmatrix} \in \mathbb{R}^{d+1} \), decode \( y \in \mathcal{R}_d(I) \) ?

Basic fact: If \( y \in \mathcal{R}_d(I) \), then \( M_k[y] \geq 0 \) is psd.

Why? If \( y = \int [x]_d \cdot dv \),

then \( M_k[y] = \begin{bmatrix} y_0 & y_1 & \cdots & y_k \\ y_1 & y_2 & \cdots & y_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_k & y_{k+1} & \cdots & y_{2k} \end{bmatrix} = \begin{bmatrix} \int x^0 \cdot dv & \int x^1 \cdot dv & \cdots & \int x^{k} \cdot dv \\ \int x^1 \cdot dv & \int x^2 \cdot dv & \cdots & \int x^{k+1} \cdot dv \\ \vdots & \vdots & \ddots & \vdots \\ \int x^k \cdot dv & \int x^{k+1} \cdot dv & \cdots & \int x^{2k} \cdot dv \end{bmatrix} \)

\( \int [x]_d \cdot [x]_d^T \cdot dv \) is psd.
TMP: Given $y \in \mathbb{R}^{d+1}$, $y \in \mathbb{R}^d$ (<i>I</i>)?

§ 3.3.1: $I = (-\infty, +\infty)$

**Theorem 3.3.2:** $y = (y_0, y_1, \ldots, y_d)$, $d = 2k$, $d = 2k+1$, assume $y_0 > 0$.

(i) $d = 2k+1$, $y \in \mathbb{R}^d (-\infty, +\infty) \iff \text{Me}[Cy] \succeq 0$ and

$$
\begin{bmatrix}
y_0 & y_1 & \cdots & y_k \\
y_1 & y_2 & \cdots & y_{k+1} \\
\vdots & \vdots & \ddots & \vdots \\
y_k & y_{k+1} & \cdots & y_{2k+1}
\end{bmatrix}
\begin{bmatrix}
p_0 \\
p_1 \\
p_k \\
p_{k+1}
\end{bmatrix}
= 
\begin{bmatrix}
y_{k+1} \\
y_{k+2} \\
y_{2k+1}
\end{bmatrix}
$$

has a solution.

(ii) $d = 2k$, $y \in \mathbb{R}^d (-\infty, +\infty) \iff \text{Me}[Cy] \succeq 0$ and

$$
\begin{bmatrix}
y_0 & y_1 & \cdots & y_{k-1} \\
y_1 & y_2 & \cdots & y_k \\
\vdots & \vdots & \ddots & \vdots \\
y_{k-1} & y_k & \cdots & y_{2k-1}
\end{bmatrix}
\begin{bmatrix}
p_0 \\
p_1 \\
p_{k-1} \\
p_k
\end{bmatrix}
= 
\begin{bmatrix}
y_{k+1} \\
y_{k+2} \\
y_{2k}
\end{bmatrix}
$$

has a solution.

(iii) Unspecified.
Proof: (i) Already know \( M_c \geq 0 \). Assume \( M_c \geq 0 \) positive definite.

Consider the linear system
\[
M_c \cdot p = \begin{bmatrix} y_{k+1} \\ \vdots \\ y_{2n+k+1} \end{bmatrix}
\]
has a solution \( p \neq 0 \).

Let
\[
p(x) = p_0 + p_1 x + \ldots + p_k x^k - x^{k+1}
\]
be a poly of degree \( k+1 \).

Then \( p(x) \) has \( k+1 \) roots \( u_1, u_2, \ldots, u_{k+1} \).

Claim: all these roots are real and distinct.

i) Suppose \( \beta = \alpha + i \gamma \) is a non-real root \( \beta \neq 0 \).

Then
\[
(x - \alpha \pm \sqrt{-1} \beta) \mid p \Rightarrow p(x) = |a_0|^2 + \beta^2 |b_0|^2, \quad b_0 \neq 0.
\]
Then \( p(x + j\beta) = 0 \) \( \Rightarrow \) \( 0 = \text{vec}(p)^T M_{k+1}[\gamma] \cdot \text{vec}(p) \)

\[
\text{vec}(a)^T M_{k+1}[\gamma] \cdot \text{vec}(a) + \beta^2 \cdot \text{vec}(b)^T M_k[\gamma] \cdot \text{vec}(b) = 0
\]

\[
\Rightarrow \text{vec}(b)^T M_k[\gamma] \cdot \text{vec}(b) = 0 \Rightarrow \text{vec}(b) = 0 \Rightarrow \beta a_1 = 0
\]

ii) \( p(x) \) has no repeated roots. Suppose otherwise, \( p \) has a repeated zero, then \( p(x) = (x - x_0)^2 \cdot q \), \( \Rightarrow \) \( p \cdot q = (x - x_0)^2 \cdot q = t^2 \).

Then \( 0 = L_y(pq) = L_y(t^2) = \text{vec}(t)^T M_k[\gamma] \cdot \text{vec}(t) \Rightarrow \text{vec}(t) = 0 \Rightarrow \text{vec}(f) = 0 \Rightarrow f = 0 \) \( \times \).
Given $y \in \mathbb{R}^{d+1}$, $y = \lambda_1 [t_1] + \ldots + \lambda_r [t_r]$, $\frac{1}{t_i} > 0$, $t_i \in \mathbb{R}$.

Algorithm 33.33: $r = \text{rank}(y)$.

1) If $r = k+1$, solve $M_k \tilde{y} - \tilde{p} = y$, $k+1: zr+1$.
2) If $r < k+1$, solve

$$
\begin{bmatrix}
    y_0 & y_1 & \ldots & y_{r-1} \\
    y_1 & y_2 & \ldots & y_r \\
    \vdots & \vdots & \ddots & \vdots \\
    y_{2k-r} & y_{2k-r+1} & \ldots & y_{2k-1}
\end{bmatrix}
\begin{bmatrix}
    p_0 \\
    p_1 \\
    \vdots \\
    p_{r-1}
\end{bmatrix}
= 
\begin{bmatrix}
    y_r \\
    y_{r+1} \\
    \vdots \\
    y_{2k}
\end{bmatrix}
$$

3) Set $p = (p_0, p_1, \ldots, p_{r-1}) \Rightarrow p(x) = p_0 + p_1 x + \ldots + p_{r-1} x^{r-1}$.
4) Find $r$ roots of $p(x) \rightarrow t_1, t_2, \ldots, t_r$.
5) Derivative coefficients $\lambda_1, \ldots, \lambda_r$. 